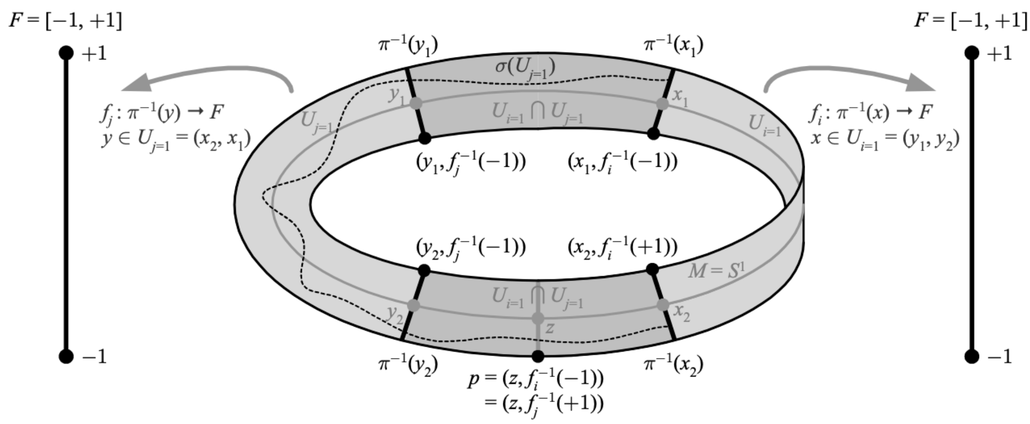
Mémoire de Master:
Histoire et philosophie des sciences et techniques (MADELHIS)

# What is geometric about gauge theories?

## History and philosophy of classical gauge theories

Réalisé dans le cadre du master MADELHIS

Année universitaire 2025-2026





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## Acknowledgements

I am deeply grateful to Andrei Rodin for agreeing to supervise this thesis. I also thank the entire teaching team of the MADELHIS Master's program for their great lectures in logic, history and philosophy of science. What I learned was invaluable to me, both as a scientist and as a person. I also thank my friends, teachers, and colleagues for the many enriching discussions on the foundations of physics.

*Peut-être Gilman aurait-il dû moins s'acharner dans ses études. Le calcul non euclidien et la physique quantique suffisent à fatiguer n'importe quel cerveau ; et quand on y ajoute le folklore, en essayant de déceler un étrange arrière-plan de réalité à plusieurs dimensions sous les allusions morbides des légendes gothiques et les récits extravagants chuchotés au coin de la cheminée, peut-on s'attendre à éviter le surmenage intellectuel ?»*

*H.P. Lovecraft, La maison de la Sorcière*

# Contents

# 1 Introduction

Gauge theories are mathematical models describing the fundamental interactions mediated by a specific type of field (the gauge fields) subject to a unique type of symmetry (gauge invariance). Contemporary physics recognizes four fundamental interactions: gravity, electromagnetism, the weak[2] and the strong nuclear forces. The latter three are undoubtedly described by gauge theories while the status of gravity as a gauge theory has been the subject of multiple debates in the past century. Beyond fundamental physics, gauge theories have also proved applicable to a great variety of contexts and have become an object of pure mathematical inquiries. However, while gauge theories unarguably represent a cornerstone of modern physics and mathematics, their philosophical status remains widely discussed and largely unclear (Belot 2002; Berghofer, François, Friederich, et al. 2023; Earman 2002; Earman 2004; Healey 2007; Leeds 1999; Redhead 2002; Rickles 2008; Weatherall 2015). This crisis in our understanding of gauge was infamously described by Redhead (2002) as ”the most pressing problem in current philosophy of physics”.

Like general relativity, gauge theories admit a formulation using the language of modern differential geometry. Within this framework, gauge fields are naturally described in terms of connections on fiber bundles, and physical field strengths correspond to the curvature of these connections. In this language, it might be tempting to interpret gauge theories as resulting from the motion of matter in curved abstract spaces, just as it is common to interpret general relativity as depicting gravity as the geodesic motion of objects in a curved space-time.

The aim of the present work is to provide an overview of the historical and philosophical foundations of gauge theories and to examine possible geometric interpretations of these theories within the framework of fiber bundle geometry.

Part I presents a historical and conceptual overview of gauge theories. This discussion will clarify how key theoretical concepts—such as gauge potentials—emerged and how their role evolved within physical theory, eventually giving rise to the philosophical puzzles associated with gauge symmetry. Part II examines the main interpretational challenges raised by modern gauge theories and reviews several proposed resolutions in the philosophical literature. Finally, Part III explores geometric interpretations of gauge theories and their philosophical implications. In particular, we will defend the view that the ontologically significant structure of gauge theories is captured by their local parallel transport properties. This perspective naturally leads to a structural realist interpretation of gauge theory according to which gauge theories describe networks of relations given by parallel transport rather than intrinsic properties attached to points.

Considering that gauge theories describe all of the fundamental interactions of nature known so far, the philosophical literature on this topic has been relatively scarce. This observation can certainly be explained by the fact that a deep understanding of gauge theories requires a rather heavy theoretical background, both in physics and mathematics. However, the very recent years have seen a growing interest in the topic, with multiple major articles and reviews – lacking before – being published in the philosophical literature while the present essay was being written (Berghofer, François, Friederich, et al. 2023; Berghofer, François, and Ravera 2025; Dougherty 2025; Gomes 2024b; Gomes 2025a; Ramírez and Teh 2025). Unfortunately, we will not attempt a detailed and pedagogical presentation of this background in the present essay, as doing so would require at least a whole book and would drive us away from the historical and philosophical points we would like to discuss. We tried to partly fill the gap with Appendices defining the key mathematical concepts and introductions to gauge theories in the language of fiber bundles can be found in Baez and Muniain (1994), Coquereaux (2002), and Schuller (2015) while more advanced reading can be found in Bleeker (1981), Hamilton (2017), Kobayashi and Nomizu (1996), Mayer (1998), and Nakahara (2003). For Clifford and geometric algebra, see Doran (1994) and Lasenby, Doran, and Gull (1998) and Benn and Tucker (1987).

[2]Electromagnetism and weak nuclear forces can be understood as a single force in the context of electroweak theory.

# Part I

# A historical and conceptual overview of gauge theories

Attempts to define, in a clear and unambiguous way, what a gauge theory is —or is not— constitute a perilous exercise. A comprehensive presentation of the standard ”textbook” approach to gauge theories in the context of particle physics can be found for example in Sec. 8.2 or in Schwartz (2013). However, such a presentation presupposes advanced knowledge of fundamental physics and is not easily accessible. Instead of confronting gauge theory head-on, we will aim to trace its contours by examining how it gradually emerged throughout modern physics.

Most of the conceptual issues with gauge theories are rooted in the notion of a **gauge field**, which generalizes the notion of **potential**. This concept is highly abstract and often constitutes a genuine source of confusion when first encountered. In a broad sense, a **gauge theory** may be characterized as a theory in which such a field is present. Understanding its meaning, and its physical implications is therefore essential.

In the following sections, we will trace the conceptual origin and evolution of the notion of potential from classical physics to contemporary gauge theories. Each section will be divided into a historical subsection and a conceptual subsection. We believe that these two approaches are complementary, as the ”potential field” is both a conceptual and a historical object. Within the sections labelled as ”history”, we keep the notation of the original authors, as we wish to study how the concepts, along with their notations and naming, evolved. Historical equations (taken from a text), are in yellow boxes in order not to be confused with modern ones. In the sections labelled as ”concepts”, we use modern notations and terminology, standardized between such sections, as we wish to clarify the concepts themselves and their inter-relations.

By examining how the status of the potential progressively shifted from auxiliary mathematical constructs to geometrically and physically indispensable objects, we aim to clarify their significance and the foundational role they play in our most fundamental descriptions of nature.

A key review on the history of electromagnetism can be found in Darrigol (2000) and for work targeting the history of gauge theories we refer to Jackson and Okun (2001), O’Raifeartaigh and Straumann (1998a), O’Raifeartaigh (1997), and Wu and Yang (2006) as well as Part III of Cao (2019).

## 2 Classical mechanics and the notion of potential

In some sense, gauge theories are theories that aim to explain the motion of objects and, as such, are rooted in mechanics. Indeed, potentials were primarily motivated as notions of mechanics[3]. In this context, they are defined as spatial integrals of forces; conversely, forces arise as spatial derivatives of potentials. Let us examine how this concept emerged in the early development of mechanics, initially as a mere computational device introduced to simplify otherwise intractable problems.

### 2.1 History of potentials and their origin in mechanics: vis viva, potential ascent and the extremal curves

In the second half of the XVII$^{\text{th}}$ century, a new mathematical description of motion emerged from the development of differential calculus by Isaac Newton and Gottfried Wilhelm Leibniz (Leibniz 1684; Newton 1687). In manuscripts dating from 1669 and culminating in the *Principia*, Newton formulated what is now called classical mechanics (Newton 1669). Its central principle is that the motion of a body is determined by forces acting upon it. A *force* is understood as a cause of acceleration, that is, of change in velocity.

Yet the conceptual roots of the notion of potential predate its mathematical formalization within Newtonian mechanics. They lie in two crucial and famous debates underlying the origin of calculus

[3]In this essay, we will not address the related notion of thermodynamic potentials.

and modern mechanics: Firstly, in XVII$^{\text{th}}$ century debates about what quantity, if any, is conserved in mechanical processes (see Hecht (2003)). And secondly in the application of differential calculus to mechanics in the second half of the XVIII$^{\text{th}}$ century to find extremal curves (see Chap. 22 of Kline (1972), Caparrini and Fraser (2013)). These two lines will converge into what will become the formulation of analytical mechanics by Lagrange in the late XVIII$^{\text{th}}$ century, based on the principle of least action.

The XVII$^{\text{th}}$ and XVIII$^{\text{th}}$ centuries saw debates about conserved quantities associated with motion. René Descartes (1644) proposed that the total "quantity of motion" in the universe is conserved. By this he meant the sum of the products of mass and speed. Although he (mistakenly) did not treat velocity as a vector quantity, this notion approximates what is now called linear momentum. Leibniz (1686) objected that Cartesian momentum does not remain constant in all collisions and introduced instead the concept of *vis viva* ('living force"), defined as the sum of the products of mass and the square of the speed. In modern terms, vis viva corresponds (up to the factor 1/2, introduced later) to kinetic energy. Leibniz showed that in elastic collisions the total vis viva is conserved. The dispute was therefore not merely terminological but concerned the correct quantitative measure of motion[4]. Momentum measures motion transferred through time (what would later be formalized as impulse), whereas vis viva measures the capacity of motion to produce effects, such as lifting a weight to a certain height. Leibniz also distinguished between *vis activa*, the active force manifest in motion, and a more latent or passive capacity to act (*potentia nuda*), from the bare potential which requires an external impulse to be transformed into a force (see Jorati (2018)). Although these notions remained partly metaphysical and were not expressed as functions over space, they introduced a crucial conceptual separation between actual motion and stored capacity for motion. In retrospect, this distinction anticipates the later separation between kinetic energy (energy of motion) and potential energy (energy associated with position/configuration). Note however here that the concept of "energy" appears only much later in the XIX$^{\text{th}}$ century.

The notion of potential makes a first appearance in the field of fluid mechanics in the work Hydrodynamica of Daniel Bernoulli in 1738 (Bernoulli 1738b)[5]. Using considerations about transformations of the vis viva, Bernoulli interprets here the motion of fluids in terms of actual descent (descensum actualem) and potential ascent (ascensum potentialem), as a possible, but yet not realized, motion[6]. Later on, Leonard Euler in his study of fluids of 1753 (Euler 1753) (p.300), introduces the function $S$ whose spatial derivatives give the components $(u, v, w)$ of the velocity in what would be called today a "potential flow" ($u = \frac{\mathrm{d}S}{\mathrm{d}x}$, $v = \frac{\mathrm{d}S}{\mathrm{d}y}$ and $w = \frac{\mathrm{d}S}{\mathrm{d}z}$)[7]. From this Euler derives an equation stating that the sum of the second spatial derivatives of $S$ must equal zero ($\frac{\mathrm{d}\ \mathrm{d}S}{\mathrm{d}x^2} + \frac{\mathrm{d}\ \mathrm{d}S}{\mathrm{d}y^2} + \frac{\mathrm{d}\ \mathrm{d}S}{\mathrm{d}z^2} = 0$). This equation is now known as the Laplace equation, and will reappear some thirty years later in the context of gravitational force (see below). However, the word "potential" does not appear to describe $S$ and it is unclear that Euler identifies it with the concept of the potential ascent of Bernoulli, even though the strong link between Euler and the Bernoulli's family is a known fact[8]. The same year of his publication of Hydradynamica, Daniel Bernoulli uses the spatial integral of the gravitational force (grauitatis vim, noted $\xi$) to express the conservation of the vis viva in order to derive the velocity $v$ associated to planetary motion going from a distance $a$ to a distance $x$ (see Bernoulli (1738a), p.117). This equation would be interpreted today as the use of the conservation of mechanical (kinetic + potential) energy. However, here again, the word potential is not used for this quantity such that the potential does not seem to be identified as a fundamental quantity of interest for itself and the parallel with his own notion of potential ascent is not done. Thus, the quantity which we would interpret today as the potential field ($\int_a^x -\xi \ \mathrm{d}x$) appears here as a useful quantity to derive the change of vis viva without being interpreted as a "potentiality".

As stated above, the other context in which the notion of potential appeared was in the study of

[4]It was later understood through d'Alembert's Traité de Dynamique (1743) that both approaches were complementary as "vis viva is the measure of a force acting through a distance while momentum is the measure of a force acting through a time." See also Iltis (1970).

[5]English translation available in Bernoulli (1968).

[6]For a discussion of Bernoulli's vocabulary see Fonteneau and Jérôme (2013).

[7]The english translation is available in Euler (1761).

[8]On Euler's contribution , as well as the one from Fontaine and Clairaut see also Cross (1983).

the elastica curve. It was indeed a major challenge of the XVIII$^{\text{th}}$ century to apply the new differential calculus in order to find solutions for curves obeying extremal properties (minimizing or maximizing a certain quantity). The most famous illustrations being the curve of quickest descent (brachystochrone) and the curve of shortest length on a surface (geodesics) (for a review see[9] Freguglia and Giaquinta (2016)). These explorations led to the joint development of multi-variable calculus and the calculus of variations. The elastica or elastic curve is the shape of a curved elastic strip. Through letter exchanges, Daniel Bernoulli indicated to Euler in 1742 that the elastica could also be understood as an extremal curve, which minimizes the "vis viva potentialis" of the curved elastic lamina (written as $\mathrm{d}s/R^2$ where $\mathrm{d}s$ is an infinitesimal arclength and $R$ is the curvature radius of the curve). Through this interpretation, the shape of the elastica comes from some vis viva stored by the elastic. Euler was then able to derive the correct equation for the elastica by extremalizing the vis viva in an appendix "additamentum I: De Curvis Elasticis" of his mostly celebrated paper (Euler et al. 1744) "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici lattissimo sensu accepti " (appendix at p.245). In the text, Euler thanks explicitly Daniel Bernoulli for his remark and refers to the vis potentialis: "On account of which, when the most celebrated and observant Daniel Bernoulli, on looking into the general nature of this problem at the highest level, indicated to me that the general force, which is present in a curved elastic lamina, can be included in a single formula he calls the force potential (vis potentialis)"[10]. While it is clear that the notion of vis potentialis in the use of both Bernoulli and Euler is vague and not a formalized notion, it is plausible that this result had some influence on Euler's view of mechanics and his understanding of the principle of least action after Maupertuis. Six years later he would publish his essay " Recherches sur les plus grands et plus petits qui se trouvent dans les actions des forces " by stating "C'est une vérité dont on ne peut plus douter, que toutes les actions, qui sont produites par les forces de la nature, renferment constamment un maximum ou un minimum"[11] (Euler 1750). This affirmation had a great future and was not contradicted since then.

The systematic and modern use of potential functions will follow, notably by Joseph-Louis Lagrange, providing a rewriting of classical mechanics based on the minimization principles discussed by Euler. While the potential theory undoubtedly appeared there in its modern form, the word itself disappeared after Daniel Bernoulli for more than a century.

In 1761, Lagrange [12] intended to follow the path set by Euler, by providing a general method in order to minimize the quantity[13] $M \int u \; \mathrm{d}s + \int M' \int u' \; \mathrm{d}s' + ...$ (where $M, M' \cdots$ are the masses of the multiple particles involved, $s, s'...$ their displacements, and $u, u'...$ their velocities). Such minimization makes it possible to solve general problems of dynamics and Lagrange provides several illustrations. This work is highly technical and Lagrange does not isolate and identify any clear function which could be identified as the potential. Similar considerations seem to apply for the work on the calculus of variations and mechanics performed in the next years.

In 1773 however, Lagrange [14] insists on a particularly useful trick consisting of considering the sum $\Sigma$ – which would today be known as the gravitational potential – given by $M/\Delta + M'/\Delta' + M''/\Delta''$ (where $\Delta, \Delta', \Delta''$ are the distances between the bodies), such that the total attraction witnessed by a body can be computed as $\frac{\mathrm{d}\Sigma}{\mathrm{d}\alpha}, \frac{\mathrm{d}\Sigma}{\mathrm{d}\beta}, \frac{\mathrm{d}\Sigma}{\mathrm{d}\gamma}$, where $\alpha, \beta$ and $\gamma$ are three orthogonal spatial directions (p.349-350). As such, without being named, $\Sigma$ is isolated as a function of peculiar interest to solve mechanical problems such as the motion of planets.

Later in 1780, Lagrange[15] writes such a function in full generality (p.24) as the integral over space of the force (similarly, but more generally to what was proposed by Bernoulli (1738a)). To our knowledge, this is the first general isolation of the potential as a quantity of interest as well as the systematic use of

[9] See also our discussion of this topic at https://leovacher.github.io/files/courbes-Vacher.pdf.

[10] Taken from the english translation by Ian Bruce available here.

[11] There is a truth that we can not doubt, that the actions, produced by the forces of nature, always contain a maximum of a minimum.

[12] Lagrange, Serret, and Darboux (1867). Tome 1: Application de la méthode exposée dans le Mémoire précédent à la solution de différents problèmes de dynamique p.365-468.

[13] Which is the "action" even though the term is not used by Lagrange.

[14] Lagrange, Serret, and Darboux (1867), Tome 6, IV: sur l'équation séculaire de la lune (p.335-399).

[15] Lagrange, Serret, and Darboux (1867), Tome 5 : XL: théorie de la libération de la Lune et des autres phénomènes qui dépendent de la figure non sphérique de cette planète (p.5-122).

the letter $V$ to describe it. Lagrange does not explain his choice of the letter $V$ for this quantity. $V$ is introduced as "Soit, pour abréger," that is "Let, for short", stressing that, in the work of Lagrange, $V$ is a mathematical object useful for computing the equations of motion, denoted by a single letter in order to simplify the derivations. Similar considerations apply to the later works of Lagrange, including his "mechanique analitique" of 1788 which will give birth to what is now known as Lagrangian mechanics in its modern form (Lagrange 1788).

$V$ will appear again extensively in the work of Pierre Simon de Laplace as a practical tool used to describe the forces generated by extended amounts of matter with various shapes, with a central interest for astronomy. In 1785, in his "Théorie des attractions des spheroïdes et de la figure des planètes"[16] Laplace says "Si l'on nomme $V$ la somme de toutes les molécules du sphéroide divisées par leur distance" with $V = \int \mathrm{d}M/r$. The notation $V$ is certainly taken from Lagrange and here again, the word "potential" is absent and $V$ is understood plainly for what it is: the sum of the all the object's particles mass divided by their distance to the center. Laplace then derived his eponymous equation in spherical coordinates. In 1798, in his famous "traité de mecanique celeste", Laplace gives a first importance status to $V$, but with a similar lack of interpretation (*"V est la somme des molécules du sphéroïde , divisées par leurs distances respectives au point attiré"* p.136) and derives again the equation which will inherit his name Eq. A (p.137): $\frac{\mathrm{d}\ \mathrm{d}V}{\mathrm{d}x^2} + \frac{\mathrm{d}\ \mathrm{d}V}{\mathrm{d}y^2} + \frac{\mathrm{d}\ \mathrm{d}V}{\mathrm{d}z^2} = 0$. This equation is the same that Euler derived thirty years before, but applied to the gravitational field instead of the velocity field of a fluid.

### 2.2 Concepts: Potentials in Newtonian mechanics

Classical Newtonian mechanics is a theory with the ambition to explain and predict the motion of physical objects. For now, we will discuss only **point mechanics**, which treats objects as points moving in space and time. While this might seem like a very crude approximation of reality, it is actually sufficient to describe a wide variety of phenomena with high precision, from the motion of planets to the dynamics of projectiles.

The fundamental proposal of Newtonian mechanics is to introduce and define a new concept, namely force, which can take different forms and is at the origin of the motion of all bodies. A force is noted $\vec{F}$, where the arrow designates that it is a vector, which can be thought of as an arrow pointing in three dimensional space. If a body is witnessing an acceleration (that is a change in speed) $\vec{a}$, it is understood as being under the action of some force $\vec{a} = \vec{F}/m$ where $m$ is a property associated to each body, called the **mass**. From this equation, we see that a more massive body will accelerate less than less massive ones under the same force. This matches our everyday intuition.

Potentials enter the picture as an object allowing to rewrite some of the forces. A **potential** $V(x, y, z)$, is a function defined over space, that is, it assigns a number to each point in space. A force $\vec{F}_V$ can be associated to such a potential by the following equation:

$$\boxed{\vec{F}_V = -q_V \vec{\nabla} V} \tag{1}$$

Here $\vec{\nabla}$ is the **gradient** operator, which is a mathematical tool allowing to compute the spatial variation of the function $V$ (for more, see Appendix A). The bigger $\vec{\nabla} V$ is, at a point, the more it varies when we move in space. Because of the arrow, $\vec{\nabla} V$ is a vector, and it points toward the direction of the variation of $V$. The minus sign in front transforms this direction into the opposite direction. $q_V$ is a property of the body called **charge**, which is another key and ubiquitous concept of gauge theory.

Equation 1, which might seem now unjustified, can be understood as the definition of the potential $V$. While not all existing forces can be associated to a potential $V$, all the forces now understood as fundamental (as gravitation and electromagnetism) can be rewritten in terms of a potential. As we saw in the historical section, the reason for their introduction and use in classical mechanics is motivated by the fact that it is much easier to solve some specific problems in terms of potentials, which are scalar quantities, than using forces, which are vectors and require complex geometric formulations.

Now, how to understand this abstraction? The simplest possible illustration of a potential, is the one of a particle in a gravitational field on earth. In that case, the charge associated to $V$ is the mass

[16] Oeuvres T.10 p.339-419, (Laplace 1799).

of the object $q_V = m$, and $V$ is a function that will increase more and more as we go up in the sky, away from the ground. Thus $\vec{\nabla}V$ will point away from the ground. Due to the minus sign, the force $\vec{F}_V$ will thus be directed downwards, and its strength will be proportional to the variation of $V$. In the presence of such a potential, the body will thus fall towards the ground as $\vec{a} = \vec{F}_V/m = -\vec{\nabla}V$.

This should be understood as follows: As its name indicates, $V$ represents a **potentiality** for the object to move when present at a point $(x, y, z)$. Furthermore, the quantity $U = q_V V$ is named the **potential energy** of the object at that point, and it represents the amount of energy that the object would gain if it was left at that point and free to move under the action of the force $\vec{F}_V$. Potentials and potential energy allow us to introduce a new modern version of mechanics where the notion of force is secondary and $V$ is enough to predict the motion of objects. This is Lagrange's analytical mechanics, based on the least action principle and arguably at the basis of all modern physics. However, we will not further describe it for now.

A natural question arises: why not consider $V$ (or $U$) as the primary "physical" object of interest in mechanics, instead of the force ($\vec{F}_V$)? After all, $V$ is a scalar function, which is mathematically simpler to handle than the vector $\vec{F}_V$. Why did the founders of mechanics not envision the world as filled with potentials $V$ that set all objects in motion? In other words, why was $V$ never promoted as the "ontological bearer" of classical mechanics? Even more striking, we saw that none of the early founders of classical mechanics explicitly named $V$ or attempted to provide a clear interpretation of its meaning.

A fundamental key to answer these questions lies in the mathematical properties of the gradient operator $\vec{\nabla}$:

$$\boxed{\vec{\nabla}(V + C) = \vec{\nabla}V} \tag{2}$$

where $C$ is any possible constant value or even any function of time $C(t)$. So the potential value can change with time as we please without any observable consequences, as long as it changes identically across all of space. This symmetry may be called shift invariance. This simple equation illustrates a problematic that will become relevant for our whole discussion and which has profound implications: two potential fields differing by a constant value, $V$ and $V + C$, produce identical forces $\vec{F}_V$ on a moving object and therefore lead to identical motion.

This leads to the common statement found in every physics textbook that energy is only defined up to an arbitrary constant; only energy differences matter, not absolute values. Concretely, as an object moves, potential energy $U$ is converted into kinetic energy. The motion depends solely on the difference between initial and final potential energy, not on its absolute value at any point. This invariance under a constant shift of $V$ is a direct consequence of its shift invariance symmetry. As such, we might dub $V$ as "unphysical", since its exact value appears to be meaningless. After a discussion of electromagnetism, we will see how this common sense idea will be challenged by recent developments in modern physics, which seriously threaten this simple view.

In particular, the key concept of **gauge fields** will absorb the potential $V$ as one of its components. The general transformation leaving the gauge field invariant, known as **gauge invariance**, will generalize the shift invariance.

# 3 Electromagnetism: thinking in terms of fields

In addition to gravity, Newtonian physics must also accommodate the electromagnetic interaction, the force governing the dynamics of charged particles. Although historically formulated within a Newtonian framework, classical electromagnetism already contains the structural seeds of what would later be recognized as a modern gauge theory. In this precise sense, Maxwellian electromagnetism can be regarded as the prototype—and the simplest non-trivial example—of a gauge theory. Understanding how this perspective emerged requires examining the conceptual status of electromagnetic potentials, from the field intuitions of Michael Faraday to the field equations of James Clerk Maxwell, and finally to the modern geometric interpretation of gauge invariance.

A central conceptual ingredient in electromagnetism is the notion of a field: a physical quantity defined at each point of space (and time). This represents a decisive shift from Newtonian mechanics, where the ontology is built from point particles interacting instantaneously at a distance. In the previous discussion, the potential $V$ was introduced merely as a scalar function over space. However, the field concept is not a mere mathematical convenience; it signals a profound reorganization of physical description. Fields possess their own dynamics and mediate interactions locally, replacing action-at-a-distance by differential equations governing spacetime-distributed quantities. This transition—from particles to fields—forms the conceptual backbone of gauge theory.

A further departure from the scalar potential framework arises when magnetic phenomena are incorporated. Classical electromagnetism cannot be formulated solely in terms of a scalar potential. The magnetic field necessitates the introduction of a vector potential $\vec{A}$, whose curl generates the magnetic field.

### 3.1 History of electromagnetism: from Faraday and Maxwell to contemporary gauge theories

The modern and systematic study of electromagnetism began sporadically between the XIII$^{\text{th}}$ and the XVII$^{\text{th}}$ centuries, with European authors such as Alexander Neckam, Peter Peregrinus de Maricourt or Gerolamo Cardano (Smith 1970) and Arabic medieval authors such as Yemeni Sultan al-Ashraf or Ibn Sim'ūn (Schmidl 2017). A major turn was then undertaken by William Gilbert in 1600 who proposed a general and extensive study of electric and magnetic phenomena (Gilbert and Wright 1600). Among other things, Gilbert understood that objects other than amber could display electric properties and that Earth behaved as a large magnet. He was followed by multiple other authors such as Robert Boyle or Otto von Guericke who proposed the first electric generator. This context gave rise to multiple experiments and discussions about the possible nature of electricity and magnetism as fluids which span the XVII$^{\text{th}}$ and the first half of the XVIII$^{\text{th}}$ century (Darrigol 2000; Heilbron 2020). After Newton's formulation of mechanics, a major challenge of the XVII$^{\text{th}}$ century was thus to insert the magnetic and electric phenomena in the language of forces within the context and language of Newtonian mechanics.

Benjamin Franklin demonstrated the electric nature of lightning with his famous experiment using a kite conducted in 1752. He is also generally recognized for the understanding of the conservation of electric charge, as well as the distinction between positive and negative charges.

It was suspected by scientists such as Daniel Bernoulli, Alessandro Volta or Franz Aepinus that the strength of the electric force could decrease as the square of the distance, just as gravity. Indeed, the force of gravity $\vec{F}_G$ was known since Newton (1687) to be proportional to the product of the mass $m_1$ and $m_2$ of the objects and the inverse of their distance $r$ squared. This question was investigated by John Robison, who never published his results, followed by Charles-Augustin de Coulomb. Using a torsion balance, Coulomb deduces that the electric force between two objects is proportional to the product of the charges divided by their distance squared (Coulomb 1785). In modern form, this law would be written for two charges $q_1$ and $q_2$ separated by a distance $r$, as[17]

$$|\vec{F}_C| = \frac{\kappa q_1 q_2}{r^2} \tag{3}$$

which shares a striking similarity with the gravitational force.

The proper formalization of the electric and magnetic forces within Newtonian mechanics happened at the beginning of the XIX$^{\text{th}}$ century through the work of Siméon Denis Poisson (1812) and George Green (1828).

While electricity and magnetism are first understood as two distinct and independent forces, it is gradually realized that the two are linked. The first major breakthrough is realized by Hans Christian Ørsted who observed the deflection of a magnetic needle by a wire carrying electric current (Ørsted 1820). Furthermore, magnetic induction was independently discovered by Michael Faraday in 1831 and Joseph Henry in 1832. Magnetic induction is the fact that a moving magnet can generate an electric current. As such, electric and magnetic phenomena can generate one another (see Hirshfeld (2006)).

[17] One generally expresses $\kappa = 1/(4\pi\varepsilon_0)$ where $\varepsilon_0$ is the vacuum's permittivity.

These new discoveries were accompanied by novel theoretical framework accounting for them. André-Marie Ampère (1826) developed a complete theory of electrodynamics accounting for a great variety of phenomena such as Ørsted's effect and ferromagnetism.

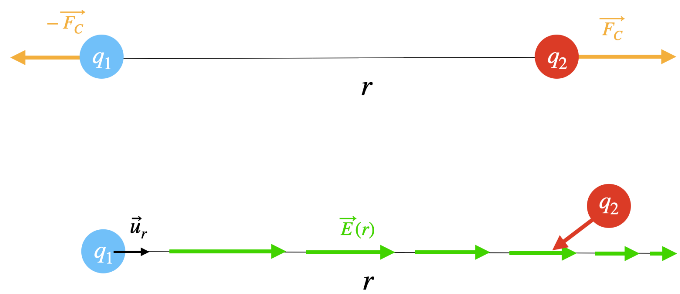
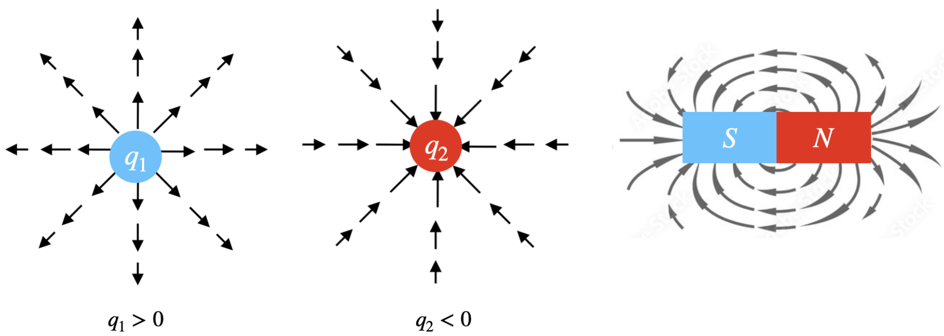



Figure 1: Top: Two particles with charge $q_1$ and $q_2$ separated by a distance $r$ interacting instantaneously through Coulomb force (here a repulsion implying that the sign of $q_1$ and $q_2$ are the same). Bottom: The presence of $q_1$ generates an electric field $\vec{E}(r)$ at each point of space labelled by $r$. If a particle $q_2$ enters the field, it will couple to the field and witness the Coulomb force.

One of the major conceptual revolutions of electromagnetism (and physics as a whole) followed, driven partly by Faraday through the emergence of the concept of fields and lines of forces. These concepts certainly originated from the study of continuous medium in fluid mechanics[18] (Euler 1736). The expressions of forces acting on objects, such as Eq. 3 seem to imply the existence of an immediate action at a distance between the two charges[19] $q_1$ and $q_2$. The nature and validity of such an action at a distance was questioned and discussed at length, already by Newton itself (see e.g. Ducheyne (2011)). However, one can interpret Eq. 3 differently by introducing the so-called **electric field** produced by the charge $q_1$ defined as

$$\vec{E} = \frac{\kappa q_1}{r^2}\vec{u}_r \tag{4}$$

where $\vec{u}_r$ is a unit length vector collinear to $r$[20]. The change of interpretation brought by $\vec{E}$ is sketched on Fig. 1. $r$ no longer represents the distance between two charges but is a continuous and infinite coordinate spanning space and having its origin on the charge $q_1$ which is placed at $r = 0$. For every $r$ one can associate a vector given by the value of the field $\vec{E}(r)$[21]. If another charge $q_2$ is added at any value of $r$, $q_2$ witnesses the force $\vec{F}_C = q_2\vec{E}(r)$. We say that $q_2$ couples to the field $\vec{E}$. As such, the existence of a single charge creates a field at every point of space to which any other charges can couple, inducing a force. The field itself is an independent entity, which can evolve and propagate, easing the tension about action at distance hidden in the notion of force.




[18]For an history of the crucial concept of fields, see e.g. Hesse (1961) (and especially the chapter VIII).
[19]From now on, focusing on electromagnetism $q_V = q$ will designate the electric charge.
[20]By reciprocity, the charge $q_2$ also produces a second electric field, such that both the charges are affected by a force.
[21]For simplicity, we illustrate here using only a single dimension of space.

Figure 2: Representations of electric and magnetic fields around a positive point charge (left), a negative point charge (center) and a bar magnet (right).

As such, in the field framework, the presence of a charge or a magnet creates a field around it as represented on Fig. 2, to which the other charges can couple. This understanding of the magnet is certainly familiar to the reader, as the presence of nearby compasses or iron dust would align with the directions of the field. Similar pictures were drawn by Faraday, who embraced the notion of field in much more than the mathematical sense, popularizing the notion that "the intervening medium must be regarded as the carrier of electric and magnetic action in a more than formal sense" (Hesse (1961) p.198). Through the concept of line of forces, Faraday thus defended that electric and magnetic interaction propagate at a finite velocity, propagating through an ambient medium called the ether (Faraday 1852). This theory is further endorsed by other authors such as William Thomson (Lord Kelvin) who proposed analogies between thermodynamic and electromagnetic phenomena (Thomson 1854) or James Clerk Maxwell in his article "On Faraday lines of forces" (Maxwell 1855). Let us however note that the concept of field proposed here differs between the authors and cannot strictly be identified with the modern concept of field (on this topic, see e.g. Gooding (1980) and Nersessian (1985)).

In parallel with all these developments, Lagrangian and Hamiltonian mechanics was formulated by numerous authors as Maupertuis, Euler, Lagrange, Poisson, Hamilton, Liouville, Darboux, Jacobi (see Goldstine (1980) and references within). This new framework further weakens the importance of the concept of force while favoring the notion of fields, as we already discussed before regarding Lagrange's formulation.

Synthesizing the work of all these previous authors armed with Faraday's conceptual legacy, Maxwell derived the final form of his set of twenty interdependent differential equations expressing the space and time evolution of the coupled electric and magnetic fields modeled by a "sea" of "molecular vortices" (Maxwell 1861). These equations were reduced to the four well known "Maxwell equations" in the modern rewriting of Oliver Heaviside (Heaviside 1893), reducing to equations equivalent to the Eq. 6 displayed in the next section. In a later work of 1865, Maxwell demonstrated – based on his equations – that light itself can be understood as an electromagnetic wave, in agreement with previous measurements done by Wilhelm Eduard Weber and Rudolf Kohlrausch in 1855 (Maxwell 1865). The equation for the so-called Lorentz force (Eq. 5) giving the most general expression of the force induced on a particle in the presence of electric and magnetic fields in the Newtonian framework was also contained implicitly in Maxwell's work. This is especially stunning as Hendrik Lorentz – who wrote its final eponymous expression in 1895 – was only 12 years old when Maxwell made such considerations. In this remarkable tour de force Maxwell thus achieved the deep unification of electricity, magnetism and light within its electromagnetic theory[22].

### 3.2 Concepts: Maxwell's equations

The **classical electromagnetism** theory provides a mathematical framework to predict and explain the observed motion of charged particles (for a review see Jackson (1998)). It requires the introduction of two time dependent vector field $\vec{E}$ and $\vec{B}$ (i.e. arrows defined in every point of space that can change with time) called respectively the **electric** and **magnetic** fields that will be the source of two new **forces** in the sense of Newton. These forces are said to be fundamental as they are believed not to be the resultant of multiple underlying processes (as friction would for example be the sum of many underlying particles hitting a body). Under the presence of these fields, a point particle will be subject to the so-called **Lorentz force**

$$\vec{a} = \frac{\vec{F}_L}{m} = \frac{q}{m}\left(\vec{E} + \vec{v} \times \vec{B}\right), \tag{5}$$

which merges the electric and magnetic forces into a single expression, quantifying the force experienced by a charged particle when moving in a space where $\vec{E}$ and $\vec{B}$ are present. This is the bridge between Newtonian mechanics, which states that acceleration is generated by forces, and electromagnetism. ×

[22]Credits should be given to Ludvig Lorenz who reached independently similar conclusions two years after in 1867 (Lorenz 1867)

is the cross product between two vectors (see Appendix A). Under the action of such a force, particles will move along the direction of $\vec{E}$ and rotate around the direction of $\vec{B}$. The force is proportional to the intrinsic property of the particle $q$ referred to as the **electric charge**, quantifying the sensitivity of the particle to the Lorentz force. The charge can be either positive or negative, impacting the direction of motion along $\vec{E}$ and the direction of rotation of the trajectory induced by $\vec{B}$.

The space and time evolution of the electro-magnetic fields is given by the Maxwell equations

$$\begin{cases} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\varepsilon_0}, \\ \vec{\nabla}\cdot\vec{B} &= 0, \\ \vec{\nabla}\times\vec{E} &= -\frac{\partial \vec{B}}{\partial t}, \\ \vec{\nabla}\times\vec{B} &= \mu_0\vec{j} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}, \end{cases} \tag{6}$$

These equations make heavy use of the $\vec{\nabla}$ operator, whose mathematical properties are recalled in Appendix A. In these equations, the charge density $\rho = \,\mathrm{d}Q/\,\mathrm{d}V$ and the charge current $\vec{j} = \rho\vec{v}$ are related to the matter properties, highlighting the interplay between charges creating fields while being impacted by them. $\varepsilon_0$ and $\mu_0$ are two constants named respectively the vacuum permittivity and permeability. Making the long story short[23], the two first equations state respectively that charge densities generate sources and sinks of the electric field, while there cannot exist any magnetic monopoles, in agreement with the sketches of Fig. 2. On the other hand, the last two equations quantify how $\vec{E}$ and $\vec{B}$ are mutually created from one another, while $\vec{B}$ can also be generated from the motion of charged particles. The second and third equations are called **charge-free (or homogeneous) Maxwell equations** while the two others are **the charged (or inhomogeneous) Maxwell equations**. Combining the Maxwell equations in vacuum, one also finds coupled wave equations for $\vec{E}$ and $\vec{B}$, propagating at velocity $c = 1/\sqrt{\varepsilon_0\mu_0}$. This is nothing less than the prediction that light is itself an electromagnetic phenomenon. Taking the Maxwell equations in a context without magnetic field $\vec{B} = 0$ and with a time independent electric field $\partial\vec{E}/\partial t = 0$, one can derive the expression of the electric force proposed by Coulomb (Eq. 3) from Eq. 6.

Taking the divergence of the last equations, and rearranging a bit, one finds $\vec{\nabla}\cdot\vec{j} = -\frac{\partial\rho}{\partial t}$ which is the continuity equation, encoding the local conservation of electric charge: all the charges entering a point have to eventually go out, they cannot be lost or created out of nowhere.

All the above equations define completely the **classical electromagnetism** theory for a point like particle in vacuum.

# 4 Potentials in classical electromagnetism

Let us now try to investigate how and why the potentials of mechanics were introduced in the context of electromagnetism, and what is their conceptual status in this theory. A surprising historical fact emerges here: unlike Lagrange in classical mechanics, Maxwell himself considered the potentials as the fundamental physical objects of his theory, instead of the fields $\vec{E}$ and $\vec{B}$.

## 4.1 History of potentials in electromagnetism

### 4.1.1 A potential for the electric field: Poisson, Green and Gauss

The first application of the concept of potential to electrostatics and the Coulomb force can be attributed to Poisson (1812). In his essay, Poisson follows exactly the same logic as Laplace but in the case of the electric field instead of the gravitational field. *"On sait que les composantes de l'attraction ou de la répulsion qu'un corps exerce sur un point donné, sont exprimées par les différences partielles d'une certaine fonction des coordonnées de ce point, savoir, de la fonction qui représente la somme des molécules du corps, divisées par leurs distances respectives au point donné: désignons donc cette somme*

[23] To reach this interpretation, one can rely on the following intuitions: The divergence of a vector field $\vec{\nabla}\cdot\vec{V}$ gives a function that quantifies at every point the difference between vectors of $\vec{V}$ entering and exciting this point. On the other hand, its curl $\vec{\nabla}\times\vec{V}$ gives another vector field indicating the torque induced by $\vec{V}$ on another vector at every point (or more intuitively, the rotation induced to a "stick" that would be carried along by the current of $\vec{V}$).

*par $V$, considérons, et cherchons la valeur de $V$ en fonction des coordonnées d'un point quelconque de l'espace."* (p.14)[24], stressing here a result that was already discussed at length by Lagrange, today summarized by the formula $\vec{F} = -q\vec{\nabla}V$. Here Poisson is able to find the expression of $V$ corresponding to the Coulomb force (Eq. 3). Poisson will later derive the so-called Poisson equation in its complete form in Poisson (1823) (p.463). This equation represents a milestone, as it generalizes the characteristic Laplace equation for potentials (which, as a reminder, was found for $S$ by Euler) on the sum of second order derivatives, to the electric potential $V$ in a case where the charge density cannot be zero.

The work of Poisson was followed by the most extensive work of George Green in 1828, in which he summarizes and extends the previous works on electromagnetism (Green 1828). Here, Green talks for the first time about *"potential function"* to describe $V$. *"[Poisson's work] are in fact founded upon the consideration of what have, in this Essay, been termed potential functions"* (p.2). This is a (late) recognition that this central quantity is important enough to deserve a name for itself. The name potential function is not related to energy considerations but most likely because $V$ can generate the force $\vec{F}$ (for more on Green's contribution, see also Grattan-Guinness (1995)). The name for $V$ became simply *"potential"* in Carl Friedrich Gauss' work in which he derived his famous theorem known as Gauss law of electromagnetism. He states *"I will give to the function V a special denomination, calling it the potential[25]"* (p.155) or in Gauss (1867a) in which he uses the same term in German. Hermann von Helmholtz (1847) is credited for the realization that the differences of the potential function are directly translatable in changes of vis viva[26].

Along with other works, the second half of the XIX$^{\text{th}}$ century is witnessing the apparition of the conservation of energy (even if the name is not coined yet). A modern reading of the concept of energy and especially the definition of "potential energy" appeared in Rankine (1853) based on thermodynamical considerations and then noted $V$[27]. Thermodynamics was indeed the main driver for the development of this notion which gradually emerged in parallel of our previous discussion. Rankine thus defines: *"Potential energy, which is measured by the amount of a change in the condition of a substance, and that of the tendency or force whereby that change is produced (or, what is the same thing, of the resistance overcome in producing it), taken jointly."* Curiously, the first use of potential energy in modern era described something else than the energy of the potential $V$, as Rankine's work does not discuss the potential functions. The word concept of potential energy to actually describe differences of potentials of moving bodies was then gradually adopted by scientists. Helmholtz in particular is credited for closing the gap between thermodynamics and mechanics especially through its concept of Spannkraft. For more discussion see e.g. Caneva (2019).

### 4.1.2 A potential for the magnetic field: introduction

While Poisson showed that, in the static case, the electric field $\vec{E}$ could be obtained by taking the spatial derivatives of a function $V$ (as $\vec{E} = -\vec{\nabla}V$), no such thing is possible for the magnetic field $\vec{B}$. However, a similar concept will exist through the notion of vector potential $\vec{A}$, from which $\vec{B}$ can be obtained as $\vec{B} = \vec{\nabla} \times \vec{A}$, where $\vec{\nabla}\times$ is an operation also involving spatial derivatives and known as the "curl" (for a presentation, see Appendix A). Moreover, the experiments of Faraday on induction showed that the presence of a time varying magnetic field can induce an electric field. This fact cannot be accounted for solely by $V$, such that the complete expression of the electric field should be $\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t$. Thus $\vec{A}$ plays the role of a "generalized potential" generating both $\vec{E}$ and $V$ from its spatial and temporal variations.

As for $V$, $\vec{A}$ will be introduced as a practical mathematical tool allowing to simplify some complex

[24] "It is known that the components of the attraction or repulsion that a body exerts on a given point are expressed by the partial differences of a certain function of the coordinates of that point, namely, the function representing the sum of the body's molecules divided by their respective distances from the given point. Let us therefore denote this sum by $V$; let us consider it, and seek the value of $V$ as a function of the coordinates of an arbitrary point in space." (personal translation)

[25] underlined in the text.

[26] Helmholtz identifies here $V$ with the electrical tension. Indeed, the study of electrokinetics advanced in parallel of mechanics and developed concepts similar to $V$ through the notion of tensions and electromotive force. It can be supposed that the link between $V$ and these notions was made before, but we will not discuss it further here.

[27] Rankine also applies his concepts to electrokinetics but not mechanics.

calculations translating the connection between electric and magnetic phenomena. However, it was quickly interpreted as a fundamental entity by Maxwell in his great unification of electromagnetism. This status was shortly discredited and criticized by some authors as Heaviside and Hertz. The use of potentials will then disappear and re-emerge as a fundamental objects within quantum theory, which will be the topic of the next chapter.

On the early history of $\vec{A}$, we refer to the excellent review of Jackson and Okun (2001), as well as Wu and Yang (2006), based on Whittaker (1951), which is however much less exhaustive regarding developments before Maxwell.

As a disclaimer, let us first stress that the modern notation using vectors in physical equations was popularized at the end of the XIX$^{\text{th}}$ century. Before, as we already saw in the previous section, equations involving vectors were mostly expressed as three scalar expressions describing the three coordinates $x, y, z$ as Euler's descriptions of fluid velocity or a single expression along infinitesimal line elements as Lagrange's integrals. As such, the modern terminology of $\vec{A}$ as a vector field could not properly emerge before these developments.

### 4.1.3 A potential for the magnetic field: Ampère, Gauss, Neumann, Weber, Kirchhoff and Helmholtz

The notion of what can be identified today as the potential vector was first considered by authors wishing to quantify the phenomenon of induction. After the groundbreaking experiment of Ørsted, a lot of work is indeed done to understand and quantify the magnetic field generated by and between electric circuits. In particular, following the study of this effect by Biot and Savart, Ampère presents complete results on magnetic fields generated by closed circuits in Ampère (1826). The first consideration of what can be understood as $\vec{A}$ were introduced less as a potential for magnetic force than as a tool for describing induction, that is how to generate an electric current through its time variations ($\vec{E} = -\partial\vec{A}/\partial t$).

The discovery of the vector potential is often attributed to Franz Ernst Neumann, who generalized Ampère's work and derived integrals expressing the current generated produced by varying magnetic field in Neumann (1845) (see also the collection of his work in Neumann (1892), edited and commented by his son). However, it is clear that, while Neumann found expressions in which $\vec{A}$ could have been lurking, he did not isolate it as a concept for itself. Ten years before, in 1835, Gauss independently reached similar considerations as well as what could be understood as a first realisation that $\vec{B} = \vec{\nabla} \times A$. Unfortunately, his work was published only later in 1867[28], and has thus very little visibility. On this contribution from Gauss on the origin of $\vec{A}$, see also Roche (1990).

Such derivations were further refined by authors as Wilhelm Weber or Grassmann and took their full maturity in the work of Kirchhoff (1857), in which the three components of an unnamed vector $U, V, W$ can clearly be found (p.530). $U, V, W$ is expressed as a sum of the integrals of the current densities $u', v', w'$ in the three spatial directions $x', y', z'$, divided by the cube of the distance $r^3$. These expressions are now understood as the components of the vector potential in a fixed gauge known as "the Kirchhoff-Weber form". While previous authors considered similar expressions, Kirchhoff clearly isolate $(U, V, W)$ as functions of interest for their ability to recover the electromotive force of components,

$$\boxed{-\frac{8}{c^2}\frac{\partial U}{\partial t}, -\frac{8}{c^2}\frac{\partial V}{\partial t}, -\frac{8}{c^2}\frac{\partial W}{\partial t},} \tag{7}$$

where $c$ is the relative velocity. This derivation is legitimately credited to Weber by Kirchhoff, but he states it here in unprecedented clarity. Furthermore, Kirchhoff investigates here relations between $(U, V, W)$ and the electric potential (noted $\Omega$), which are true only in his choice of gauge and would be written in modern notation as $\vec{\nabla} \cdot \vec{A} = \partial V/\partial t$. Following these works, Helmholtz publishes a series of work in which he attempts to understand the equivalence of multiple expressions considered for $\vec{A}$ in the literature (Helmholtz 1870; Helmholtz 1872; Helmholtz 1873; Helmholtz 1874). These first investigations will be the precursor of a discussion of modern gauge invariance.

[28] Gauss (1867b) p.609 "Das Inductionsgesegtz".

### 4.1.4 A potential for the magnetic field: Thomson and Maxwell

In parallel, $\vec{A}$ was discovered from its capacity to generate $\vec{B}$ through its spatial derivatives ($\vec{B} = \vec{\nabla} \times A$) from more theoretical perspectives through the study of exact differentials. In 1847, William Thomson considered how expressions of the form $X \, \mathrm{d}x + Y \, \mathrm{d}y + Z \, \mathrm{d}z = \mathrm{d}f$ can be used to give a "mechanical model" of the electric, magnetic and galvanic[29] forces (Thomson 1847). This work is inspired from the previous study of George Gabriel Stokes of 1845 in which similar considerations were used to describe elastic solids (Stokes 1845). These considerations will lead to the derivation of what is now known as the "Stokes theorem" by Thomson, Stokes and independently by Hermann Hankel (see Katz (1979)).

The goal of Thomson (1847) is thus to find how the triplet $X, Y, Z$, interpreted as the components of the force vector, can be derived as the exact differential of a unique scalar function $f$. It was this exact consideration which already led previous authors to consider the potential $V$ in the case of gravity and the electric force. For the magnetic force, Thomson considers the expression (p.78)

$$\boxed{\left(\frac{\mathrm{d}\beta}{\mathrm{d}z} - \frac{\mathrm{d}\gamma}{\mathrm{d}y}\right) \mathrm{d}x + \left(\frac{\mathrm{d}\gamma}{\mathrm{d}x} - \frac{\mathrm{d}\alpha}{\mathrm{d}z}\right) \mathrm{d}y + \left(\frac{\mathrm{d}\alpha}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}x}\right) \mathrm{d}z = \mathrm{d}\frac{lx + my + nz}{r^3}.} \tag{8}$$

for three scalar functions $\alpha, \beta, \gamma$ – which will become the three components of the potential vector $\vec{A}$ – and can solve the above equation for the expressions (Eq. (3) p.78):

$$\boxed{\alpha = \frac{mz - ny}{r^3}, \quad \beta = \frac{nx - lz}{r^3}, \quad \gamma = \frac{ly - mx}{r^3} \ldots\ldots} \tag{9}$$

In that case, just as for the electric force, the three term of the differential can be understood as the three components $X, Y, Z$ of the magnetic force exerted on a unit magnet by a small magnet of unit moment, placed at the origin, with axis of polarization in the direction $l, m, n$ (Eq.II p.79):

$$\boxed{X = \frac{\mathrm{d}\beta}{\mathrm{d}z} - \frac{\mathrm{d}\gamma}{\mathrm{d}y}, \quad Y = \frac{\mathrm{d}\gamma}{\mathrm{d}x} - \frac{\mathrm{d}\alpha}{\mathrm{d}z}, \quad Z = \frac{\mathrm{d}\alpha}{\mathrm{d}y} - \frac{\mathrm{d}\beta}{\mathrm{d}x} \ldots\ldots} \tag{10}$$

The above expression can be identified as $\vec{B} = \vec{\nabla} \times \vec{A}$ in modern notation, with $\vec{B} = (X, Y, Z)$ and $\vec{A} = (\alpha, \beta, \gamma)$. However, Thomson does not name nor isolate the triplet $(\alpha, \beta, \gamma)$ as the quantities of interest but rather the function of $f = (lx + my + nz)/r^3$ which he identifies as the "potential due to a small magnet". Thomson will later re-use these concepts in a more general study of magnetism in 1850 (Thomson 1850) in which the components of what can be identified as $\vec{A}$ will also be named $F, G, H$ as the integral quantities appearing in the context of magnetic induction (see previous subsection).

Against the generally admitted view that potentials are simple mathematical artifacts, Maxwell used potentials to derive his equations and considered them gradually as the fundamental entities describing the electric and magnetic fields (on this topic see Bork (1967)). As such, he is the one to be credited for most of the modern terminology and views regarding the potentials. In Maxwell's work, the vector potential will gradually emerge as a mathematical translation of the concept of "electro-tonic state" proposed by Faraday. Here again, following Thomson, Maxwell seeks for a mechanical description of the electro-tonic state, following the work on elastic solids and of the motion of viscous fluids performed by Stokes and others.

Following Thomson, Maxwell first notes $\alpha_0$, $\beta_0$ and $\gamma_0$ for the three components of $\vec{A}$, which he names "Electro-tonic functions", or "components of the Electro-tonic intensity" in Maxwell (1855).

His views evolved in Maxwell (1861) and further in Maxwell (1865), in which he develops his full theory of electrodynamics. In these works, the potentials are clearly the relevant quantities for the dynamics. The electromagnetic potential is now noted $\Psi$ while the components of the vector potentials are now noted $F$, $G$ and $H$ and named the 'electromagnetic momentum'. This choice of

[29] generation of electric current by chemical action.

name is particularly interesting and is based on the analogy between the relationship of force and momentum ($\vec{F} = \mathrm{d}\vec{p}/\mathrm{d}t$) and the relationship of the electric field generated by induction and the potential vector ($\vec{E} = -\partial\vec{A}/\partial t$). Combining the two equations together we understand $\vec{A}$ as a change of momentum per unit of charge. This terminology will disappear but has conceptually a great future in quantum mechanics as we will see in the next chapters.

Along with this change of terminology and formalism, the conceptual role accorded to the potentials accordingly shifts, such that Maxwell's interpretation of his work seems to change from a mechanical model to a model in which the fields and their energy are the fundamental quantities of interest. Maxwell will further develop his views in his complete treatise of 1873 which will become a major reference for electrodynamics (Maxwell 1873). In this extensive work, Maxwell seeks a potential for the magnetic field. To do so, Maxwell uses Hamilton's quaternions in a way that is now rarely used, but which allows him to clearly put forward the parallels between what we have noted $\vec{A}$ and $V$. Indeed, after considering some specific cases in which a scalar potential exists for the magnetic force, Maxwell considers the use of a "vector-potential" $\mathfrak{A}$ allowing to express the magnetic induction $\mathfrak{B}$ as[30][31]

$$\boxed{\mathfrak{B} = \nabla\mathfrak{A}} \tag{11}$$

from which he concludes that "the magnetic induction is the curl of its vector potential."[32] He also further identifies of electromagnetic momentum $(F, G, H)$ (which he calls here the 'electrokinetic momentum') and the vector potential $\mathfrak{A}$, rendering the two concepts indistinguishable. Considering the major influence of his Treatise, we can attribute the generalization of the letter $A$ as well as the name "vector-potential" to Maxwell[33].

Furthermore, he considers the transformation[34]:

$$\boxed{\left.\begin{aligned} F &= F' - \tfrac{\mathrm{d}\chi}{\mathrm{d}x} \\ G &= G' - \tfrac{\mathrm{d}\chi}{\mathrm{d}y} \\ H &= H' - \tfrac{\mathrm{d}\chi}{\mathrm{d}z} \end{aligned}\right\}} \tag{12}$$

for an arbitrary function $\chi$ and continues by saying that "the quantity $\chi$ disappears from the equations [Eq. 11][35] and it is not related to any physical phenomenon." As such, he generalizes Helmholtz's work by presenting the modern equation of gauge invariance for $\vec{A}$ in full generality (but lacking the co-transformation of $V$).

Considering all the results presented in his Treatise, the modern understanding of $\vec{A}$ (and $V$) is pretty much set after Maxwell's contribution. While Maxwell considered the scalar potential as "a mere scientific concept" and that there were "no reason to regard it as denoting a physical state", he undoubtedly regarded the electro-tonic state (vector potential) to have a great significance, considering that it "may even be called the fundamental quantity in the theory of electromagnetism" (see Dougherty (2025) and references therein).

The modern formulation of Maxwell's equations in terms of $\vec{E}$ and $\vec{B}$ given by Eq. 6 is due to the later contribution of Oliver Heaviside, who led a vendetta against the use of potentials (Heaviside 1893). Heaviside refers to them as "parasites" which need to be "murdered" for "a great gain in definiteness and conciseness" (see Wu and Yang (2006) and references therein). After Heaviside's reformulation of electromagnetism –considered equivalent and more practical than Maxwell's – $\vec{A}$ and $V$ largely disappeared from the literature untill the advance of quantum mechanics.

[30] Maxwell (1873) Volume 2, p.32, note 406.

[31] To get this simple expression highlighting the similarity between the scalar and the vector potentials, Maxwell not only uses the properties of quaternions but also what we would consider as the specific gauge condition $\nabla \cdot \vec{A} = 0$.

[32] As a sidenote, according to Bork (1967), the word "curl" is also an invention of Maxwell.

[33] Note that while $\mathfrak{A}$ might resemble the latin letter $V$, it is actually corresponding to the letter $A$ in the so-called "Fraktur" font.

[34] Maxwell (1873) Volume 2, p. 256, note 617, equation (7).

[35] In the text Maxwell refers to the component form of Eq. 11 that is $a = \frac{dH}{dy} - \frac{dG}{dz}, b = \frac{dF}{dz} - \frac{dH}{dx}, c = \frac{dG}{dx} - \frac{dF}{dy}$ which can be found in Volume 2, p. 233, note 591, equation (A).

### 4.2 Concepts: potentials, gauge invariance and gauge fixing in electromagnetism

We now turn to the introduction of potentials in classical electromagnetism using modern notations. One might initially expect a straightforward generalization of the Newtonian construction: introduce two scalar fields, $V_E$ and $V_B$, such that the electric and magnetic forces can both be recovered from a relation of the form given in Eq. 1. While this strategy succeeds for the electric interaction, it fails for the magnetic one.

A superficial reason is immediately apparent: unlike the electric case, there is no independent magnetic charge $q_B$ that would play the role of a source in a force law analogous to Coulomb's law. However, the impossibility is more fundamental than the observed absence of magnetic charge. The magnetic contribution to the Lorentz force, (Eq. 5), depends explicitly on the velocity. A force derivable from a scalar potential must be expressible as the gradient of a scalar function depending only on position. It is simply impossible to express the magnetic force as the gradient of a scalar potential (unless one uses quaternions or other more advanced mathematical tools, as was the case for Maxwell as discussed in the previous section). Thankfully, not all hope is lost. One can always express $\vec{E}$ and $\vec{B}$ in terms of two new fields $V$ and $\vec{A}$ called the **electric potential** and **vector potential** as

$$\begin{cases} \vec{E} &= -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}, \\ \vec{B} &= \vec{\nabla} \times \vec{A}. \end{cases} \tag{13}$$

This appears to be the proper way to generalize Eq. 1 to the case of electromagnetism. However, we are not able to see the true depth in this construction yet. For this, we will have to wait for the advent of special relativity.

Let us now see how Eq. 2 generalizes in our case. Since the physically relevant quantities $\vec{E}$ and $\vec{B}$ are expressed as derivatives of these new fields, only their difference should matter (as e.g. energy). Indeed if one redefines $V' = V + C$ or $\vec{A'} = \vec{A} + \vec{C}$, simply shifting their values by a constant number/vector, $\vec{E}$ and $\vec{B}$ remain the same and so should all the predictions of classical electromagnetism. Then Eq. 2 and its interpretations still apply here. However, the potentials can be transformed in much more general ways which would leave the physics untouched. Considering any smooth function $\Lambda(t, x, y, z)$, all the observable phenomena remain unchanged under the transformation[36]

$$\begin{cases} V &\to V' = V + \frac{\partial \Lambda}{\partial t}, \\ \vec{A} &\to \vec{A'} = \vec{A} - \vec{\nabla}\Lambda, \end{cases} \tag{14}$$

this is a **gauge transformation** in the context of electromagnetism. Indeed, under these transformations

$$\begin{cases} \vec{E} &\to \vec{E'} = -\vec{\nabla}V + \vec{\nabla}\frac{\partial \Lambda}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\vec{\nabla}\Lambda = \vec{E}, \\ \vec{B} &\to \vec{B'} = \vec{\nabla} \times \vec{A} - \vec{\nabla} \times \vec{\nabla}\Lambda = \vec{B}, \end{cases} \tag{15}$$

that is $\vec{E}$ and $\vec{B}$ are invariant (remain the same) under a gauge transformation. We used here the commutation of derivatives $\vec{\nabla} \times \frac{\partial}{\partial t} = \frac{\partial}{\partial t}\vec{\nabla}\times$ and the fundamental properties of $\vec{\nabla}$: $\vec{\nabla} \times \vec{\nabla} f = 0$ (for a field $f$).

Hence, since it is possible to change the values of $\vec{A}$ and $V$ without changing the physical predictions of the theory, the exact values of the gauge fields can be understood as pure mathematical objects without any physical content, just like $V$ was in Newtonian mechanics. This is the essence of the name "gauge transformation", understood as a rescaling, like one could change the space between two ticks of a ruler, i.e. change units, without changing the size of the system under consideration.

Hence it should always be possible, for convenience, to pick an arbitrary choice of $V$ and $\vec{A}$, through the process known as *gauge fixing*, in a such a way that the calculation simplifies, without changing the validity of the results obtained on $\vec{E}$ and $\vec{B}$.

Let us note here the following curious fact: if one were to consider $V$ and $\vec{A}$ as the fundamental fields of the theory instead of $\vec{E}$ and $\vec{B}$, the two Maxwell equations that do not involve $\rho$ and $\vec{j}$ (charge-free)

[36] Here and in the remainder of this work, primes will denote quantities after a gauge transformation.

comes naturally from their definition. Indeed

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0, \\ \vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(-\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times A) = -\frac{\partial \vec{B}}{\partial t}. \end{cases} \tag{16}$$

where we used again some properties of $\vec{\nabla}$, as $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$ for $\vec{v}$ a vector field. While this fact can appear either trivial or confusing right now, it has deep geometrical origins as we will see later.

# 5 A plot twist: Potentials in quantum physics: the Aharonov-Bohm effect and the gauge argument

So far, the conceptual role of the potential fields is fairly straightforward. We realised mostly that $V$ and $\vec{A}$ were introduced within the classical theory of electromagnetism as mathematical tools to simplify the calculations of $\vec{E}$ and $\vec{B}$. We also saw that, since only the derivatives of $V$ and $\vec{A}$ are relevant for the physics, it is possible to change their values without changing the predictions of the theory. Hence, we are tempted to conclude that $V$ and $\vec{A}$ are pure mathematical objects without any physical content. However, we miss two crucial revolutions that will completely change the role played by potentials and drastically complicate this picture: the advances of quantum mechanics and special relativity.

By looking at how the gauge fields appear within the quantum theory of a single particle, we will be forced to ponder our first interpretation of gauge as mathematical surplus. We will indeed have to conclude that, one way or another, **the presence of the gauge fields can have observable consequences**. Indeed, within a quantum mechanical context, we will see that it is possible to create experiments in which the field $\vec{A}$ can impact the behavior of electrons, even in regions in which the value of both $\vec{E}$ and $\vec{B}$ are zero. We will then have to weigh carefully all the philosophical consequences of this puzzling phenomenon, known as the **Aharonov-Bohm effect**.

## 5.1 History: making sense of electromagnetism in a quantum theory

### 5.1.1 The birth of quanta, an electromagnetic story

The historical origins of quantum theory lie in the study of electromagnetic radiation, especially the thermal radiation emitted by heated bodies.[37] By the end of the nineteenth century, the spectrum[38] of thermal radiation was well characterized experimentally, though its origin remained obscure. Kirchhoff (1857) laid the foundations of black-body theory by introducing the idealized "black body," which absorbs all incident radiation and whose emission at a given wavelength depends only on its temperature $T$. Stefan (1879) inferred empirically that the total emitted energy scales as $T^4$; Boltzmann (1884) later derived this thermodynamically, yielding the Stefan–Boltzmann law. In 1893, Wien derived the wavelength of maximal emission, now known as Wien's displacement law, and in 1896 proposed a formula for the full spectrum (Wien 1896). Lummer and Pringsheim soon showed, however, that Wien's law agreed with experiment only at short wavelengths, failing at larger wavelengths and higher temperatures. Rayleigh (1900), treating radiation as standing electromagnetic waves in a cavity analogous to sound waves, proposed a different formula, which Lummer and Pringsheim showed to be valid only at large wavelengths.

The major breakthrough of the understanding of the black-body radiation jointly giving birth to quantum physics was done in 1900/1901 by Max Planck, somewhat unwillingly[39] (Planck 1901). From 1897 to 1899, Planck investigated Wien's law, assuming it as valid, and further investigated the thermodynamical consequences of this law for the black-body system assuming that the body is itself

[37] For a condensed history of thermal radiation, see Mavani and Singh (2022); for broader reviews, see e.g. Whittaker (1953).

[38] i.e. the energy emitted at each electromagnetic frequency.

[39] The fine details of the history of Planck's derivation are yet still subject to debate due to the loss of most of Planck's original work during the second world's war. In the present essay, we base ourselves on the detailed historical account of Planck's hypothesis proposed by Nauenberg (2016).

composed of an ensemble of so-called Hertzian electromagnetic oscillators in the context of Maxwell's electromagnetism. In particular, Planck was investigating the dependence of the entropy of the system with its energy. After being aware of the disfavor of Wien's law for small wavelengths, Planck derived another phenomenological law based on his previous works which was both in agreement with Wien at small wavelengths and new measurements at larger wavelengths. This new law was then further validated at all wavelengths by increasingly precise measurements. After the discovery and validation of his formula, Planck devoted his time to find a physical way to understand and interpret it from statistical considerations, based on Boltzmann's work. He quickly came to the conclusion that the energy of the oscillators composing the black-body can only have a discrete energy proportional to the frequency of the radiation as $\varepsilon = nh\nu$ where $h$ is the "elementary quantum of action" which is now known as Planck's constant. While such a discreteness for the energy cannot be explained within classical electromagnetism, Planck clearly considered it as a phenomenological but powerful relation, not as a change of paradigm. His derivation turned out to be the first application of quantum physics which will lead him to the Nobel prize in 1919. Planck will still remain in disagreement and opposition with the later developments of quantum mechanics for the remainder of his life.

After Planck's derivation, most of the physicists, including Planck himself; were uncomfortable with the seemingly required assumption about energy discreteness that could not be justified from first principles. After a correction suggested by James Jeans (1905), Rayleigh (1905) proposed a more thorough and refined derivation of his law based on classical electromagnetism . The resulting law, known as the Rayleigh-Jeans law remains valid at large wavelength but dramatically diverge for small wavelengths. This failure of classical electromagnetism is now known as the "ultraviolet catastrophe"[40] and is commonly presented as the original motivation for quantum theory. Let us however stress again here that Planck was clearly not aware of the ultraviolet catastrophe and his work was not a reply to this problem of classical physics. The consensus that classical electromagnetism could not explain fully the black-body radiation required a significant amount of time and the later developments of quantum theory.

In 1905, Albert Einstein realized that taking Planck's hypothesis seriously could illuminate phenomena beyond black-body radiation (Einstein 1905a). He argued that electromagnetic radiation behaves as a collection of "mutually independent energy quanta," and used this idea to explain the photoelectric effect, photoluminescence, and the ionization of gases by ultraviolet light[41]. Einstein thus accepted that light could display non-classical, particle-like properties and explored their consequences. His work on the photoelectric effect had a major impact and led to his only Nobel Prize, in 1921. The photoelectric effect had been discovered by Hertz (1887), and was later understood as the ejection of microscopic particles—electrons—from metals exposed to sufficiently energetic light. Lenard (1902) showed experimentally that the energy and number of emitted electrons were not proportional to the intensity of the incident light, in direct conflict with classical electromagnetism. Einstein explained this by using Planck's hypothesis: the relevant quantity is the frequency of the incident light, not its intensity.

The discreteness of light and of its interaction with matter became the central topic of the famous 1911 Solvay Congress, entitled "The theory of radiation and quanta," which raised many of the questions that would shape subsequent work. In 1913, Niels Bohr explained the line structure of the hydrogen absorption spectrum by proposing an atomic model in which the angular momentum of electrons can take only discrete values along their orbit around the nucleus (Bohr 1913). This suggested that the phenomenological procedure of discretizing otherwise continuous classical systems might have applications far beyond radiation itself.

A first interpretative framework was proposed by Louis De Broglie (1924) in his doctoral thesis. Using considerations about relativistic waves, De Broglie proposed that a material particle of momentum $p$ can be associated with a wave of wavelength $\lambda = h/p$. This gave a way to understand Bohr's model by allowing only standing-wave orbits. Three years later, two independent teams led by George Paget Thomson and Clinton Joseph Davisson demonstrated experimentally that electrons do behave as waves, exhibiting diffraction and interference (Davisson and Germer 1927; Thomson 1927). Together

[40]The name was given much later by Paul Ehrenfest (1911).

[41]On Einstein's contributions to quantum mechanics, see Singh (2005).

with Einstein's earlier discussion of light quanta, De Broglie's hypothesis forms the other half of the so-called "wave–particle duality" of quantum mechanics, according to which every quantum object can display both particle-like and wave-like properties. These breakthroughs led to De Broglie's Nobel Prize in 1929 and to the shared Nobel Prize of Thomson and Davisson in 1937.

A major effort then began to provide a consistent mathematical formulation of these phenomena, giving rise to modern quantum mechanics. The first nearly complete and consistent formulation was the matrix theory developed by Werner Heisenberg, Max Born, and Pascual Jordan between 1925 and 1927. Matrix mechanics uses time-dependent matrices to represent observables such as momentum or energy and to compute transition probability amplitudes. The non-commutation of certain matrices encodes essential features of quantum systems, including what would become the Heisenberg uncertainty principle. Although extraordinarily powerful predictively, this formalism marked a sharp increase in the abstractness of the theory. Heisenberg (1925) first proposed the original formulation while seeking a better account of the hydrogen spectrum than Bohr's model could provide. Born and Jordan (1925) then extended and refined it, and the formalism reached its final form in the 1926 paper by the three authors (Born, Heisenberg, and Jordan 1926). On this basis, Wolfgang Pauli carried out the complete calculation of the hydrogen spectrum in 1926 (Pauli 1926).

In 1926, Erwin Schrödinger proposed a different formulation of quantum mechanics in terms of waves—"wave mechanics"—evolving under his eponymous equation (Schrödinger 1926a). Although this formulation seemed to lessen some of the apparent strangeness of the new theory, Born showed the same year that Schrödinger's wave is best interpreted as a probability amplitude for observing the particle at a given point in space—the so-called Born rule (Born 1926). Later that same year, it was shown that Schrödinger's wave mechanics could be embedded within the matrix formalism (Schrödinger 1926b). On this topic, see Perovic (2008). The appearance of irreducible probabilities in quantum mechanics prompted deep and enduring debates about its foundations, debates that continue to this day.

Up to this point, the notions of scalar and vector potential seem almost entirely absent from debates on the early foundations of quantum theory, following their careful elimination from classical electromagnetism by Hertz and Heaviside. We now turn to how $V$ and $\vec{A}$ made a sudden comeback in quantum mechanics in the 1920s.

### 5.1.2 Inserting the potential vector in a quantum theory

Curiously, while quantum mechanics was originally motivated by some properties of light and the electromagnetic field, it could not account properly for the electromagnetic phenomenon until much later, through the advancement of quantum field theory. The reason for this is due to two crucial characteristics lacking in standard quantum theory: a framework accounting for special relativity, as light (photons) travel in space with the speed of light $c$ and are by definition the supreme relativistic objects and multiple particle treatment (second quantization) because the understanding of major electromagnetic phenomena relies on the fact that photons are constantly absorbed and re-emitted. How to address these points is left for the next sections.

However, even if it was derived much later, a first description of electromagnetism in a quantum context is possible by considering the influence that a classical electromagnetic field can have on the wavefunction of a quantum particle. We will assume the correct expression for now, and investigate on its physical consequences.

## 5.2 Concepts: quantum interference and the Aharonov-Bohm effect

### 5.2.1 Schrödinger equation, wavefunctions and quantum interference

To avoid drowning ourselves in the mathematical details of quantum mechanics, we will for now gloss over the most basic concepts and results of quantum mechanics that are required to understand the role of potentials in quantum physics. For a pedagogical introduction, see e.g. Susskind and Friedman (2014) while we refer to Cohen-Tannoudji, Diu, and Laloë (1977) for a more advanced reading (see also the very dense but clear introduction in Ismael (2021)).

Schrödinger's wave mechanics is based on the idea that the particles of Newtonian mechanics should be described by complex functions $\psi(x,t)$ called the wavefunctions. In its standard formulation, quantum mechanics is intriniscally probabilistic and can thus only answer questions like "what is the probability that ....?". The probability to observe a particle at position $x$ will be given by $P = |\psi|^2 = \psi^*\psi$, where the star is the complex conjugate. In abscence of forces, the evolution of the wavefunction in time is given by the **Schrödinger equation** for the free-particle

$$\boxed{\mathbb{i}\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi} \tag{17}$$

with $\mathbb{i}^2 = -1$ the imaginary unit and $\hbar$ the reduced Planck constant. Solving this equation gives solutions of the form

$$\psi \propto e^{\mathbb{i}(\vec{p}\cdot\vec{x}-Et)/\hbar} \tag{18}$$

representing oscillating plane waves. Thus, the regions were it is more or less probable to find a particle when measuring are oscillating. As such, the wave-functions of free particles displays an oscillatory behavior both in $x$ and $t$. This is the wave/particle duality. The wave-vector and the pulsation of the wave is related to the momentum and the energy of the particle respectively, through the de-Broglie relations $\vec{k} = \vec{p}/\hbar$ and $\omega = E/\hbar$.

Just as sounds or water waves, two different wavefunctions can interact, possibly producing some **interferences**. As Schrödinger equation is linear, the sum of two solutions is also a solution and hence, the addition of two wavefunctions is also a physical solution. But because wavefunctions are complex functions, two wave functions $\psi_1$ and $\psi_2$ obey the property $|\psi_1+\psi_2|^2 = |\psi_1|^2+|\psi_2|^2+\psi_1^*\psi_2+\psi_1\psi_2^* \neq |\psi_1|^2+|\psi_2|^2$ and thus the probability to find any particle at a point $P(x) = |\psi_1+\psi_2|^2$ will not be simply given by the sum of the two independent probability of each individual wavefunction $|\psi_1|^2$ and $|\psi_2|^2$. This fact was used by Thomson (1927) and Davisson and Germer (1927) to put forward the wave properties of matter proposed by Broglie (1924). One of the most remarkable application of quantum interference is given by the so-called "double slits experiments". Consider a free electron traveling in space. Its wavefunction will thus be given by the plane wave solution of Schrödinger's equation (Eq. 18). If this wave encounter a screen with two small slits separated by a distance $d$, the wavefunction will be split in two independent ones $\psi_1$ and $\psi_2$ associated with each split. If a screen is placed at a distance $l$ of the slits, the final wavefunction on the screen will be given by $\psi_1+\psi_2$ on the screen. It is possible to show that

$$\boxed{P(\tilde{x}) = |\psi_1+\psi_2|^2 \propto \cos\left(\frac{\pi d}{\lambda l}\tilde{x}\right)^2} \tag{19}$$

where $\tilde{x}$ is the position on the screen labelled by the distance from its center.

### 5.2.2 Quantum treatment of the Lorentz force and gauge invariance

In the presence of an electric $\vec{E}$ and magnetic field $\vec{B}$, the Schrödinger equation for a particle of charge $q$ and mass $m$ will be modified. This is expected, as the particle must witness Lorentz force, which will change its probability to be at one point or another. Without further justification, we will accept that the Schrödinger equation gets modified as follows[42]

$$\boxed{\mathbb{i}\hbar\frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-\mathbb{i}\hbar\vec{\nabla}-q\vec{A}\right)^2\psi + qV\psi} \tag{20}$$

A surprising fact is that, in this equation, the electromagnetic fields $\vec{E}$ and $\vec{B}$ are completely absent. The only electromagnetic quantities appearing in the equation are the potentials $V$ and $\vec{A}$. Even more, it is simply impossible to write the Schrödinger equation in terms of $\vec{E}$ and $\vec{B}$ without using $V$ and

[42]This approach is semi-classical as it treats the particle as a quantum object while the EM field is still considered as a classical object (instead of a proper operator). A full quantum treatement of classical electromagnetism will only be possible in the framework of quantum electrodynamics, which we keep for later.

$\vec{A}$. This should be compared with the equation of motion in classical electromagnetism, which can be expressed solely in terms of $\vec{E}$ and $\vec{B}$ (Eq. 5) This is a first hint that, within quantum mechanics, the potentials are not just mathematical tools but have a deeper role that they used to play. As can be quickly seen, the rule to go from the free Schrödinger equation (Eq. 17) to the one in the presence of an electromagnetic field (Eq. 20), it to proceed to the replacement of the gradient and the time derivative using the rule known as **minimal coupling**:

$$\begin{cases} -\mathbb{i}\hbar\vec{\nabla} \to -\mathbb{i}\hbar\vec{\nabla} - q\vec{A}, \\ \mathbb{i}\hbar\frac{\partial}{\partial t} \to \mathbb{i}\hbar\frac{\partial}{\partial t} - qV. \end{cases} \tag{21}$$

In quantum mechanics $-\mathbb{i}\hbar\vec{\nabla}$ is the operator associated to the **momentum** of the particle while $\mathbb{i}\hbar\partial/\partial_t$ is associated to its **energy**[43] (Hamiltonian). We see that, just like in classical mechanics, the quantity $U = qV$ plays the role of a (potential) energy while $q\vec{A}$ is contributing to momentum, just as Maxwell's terminology was anticipating.

Now, one might wonder how the gauge invariance appears in this framework. To see this, consider a new state $\psi'$ built from the original one as

$$\psi' = e^{\frac{-\mathbb{i}q\Lambda(x,t)}{\hbar}}\psi \tag{22}$$

with $\Lambda(\vec{x}, t)$ any function of space and time. Replacing $\psi$ by $\psi'$ in Eq. 20 and rearranging the equations, one finds back the exact same equation but with $V$ and $\vec{A}$ replaced by $V'$ and $\vec{A}'$, which are the gauge transformed potentials of Eq. 14. Hence $\psi'$ evolves through the same Shrödinger equation as $\psi$ but with the Gauge transformed potentials $\vec{A}'$ and $V'$. Put the other way around: for the EM physics to be invariant under a gauge transformation given by $\Lambda(x, t)$, the wavefunction must also transform[44] under a phase transformation as $\psi \to \psi'$. It can then be showed that such transformation also leaves all the predicted mean values for measurable quantities (observable) invariant.

Note here that the free Schrödinger equation (Eq. 17) is not invariant under Eq 22 (due to the derivatives acting on $\Lambda(x, t)$). One can then understand the presence of the gauge fields (the minimal coupling rule) as a way to restore the invariance of the theory under local phase shift of the wavefunction. Put it in another more engaging way (often presented as such in classical textbooks): **fundamental interactions between particles emerge from gauge symmetries**. This idea, known as the **gauge argument** or **gauge principle** seems to represent a pivotal turn in our understanding of gauge fields. For a more detailed presentation of this argument in the context of classical quantum mechanics see Barlow (1990). For a discussion of the significance of the gauge argument, see Sec. 10.

As such, the gauge transformation of classical electromagnetism (Eq. 14) are still present at the level of quantum mechanics, leaving the physics invariant by asking for a phase shift of the wavefunction. Note again that, while we can still interpret here the potentials as mathematical surplus, the electromagnetic fields $\vec{E}$ and $\vec{B}$ are completely absent from the formulation. We will now see that, besides being central for the formulation, $\vec{A}$ and $V$ seems to play a deeper role within quantum mechanics.

#### 5.2.3 The (magnetic) Aharonov-Bohm effect

Without any doubt, the most important turn regarding the interpretations of gauge theories is given by the discovery of the so-called Aharonov-Bohm (AB) effect. Applied to electromagnetism, the AB effect proves unambiguously that the presence of a potential vector $\vec{A}$ in a region in which electrons transit can have observable consequences even when the magnetic field $\vec{B}$ is strictly zero in that region.

The discovery of the AB effect is much more modern than the formal developments of quantum mechanics discussed in the previous sections. High chances are that this effect was already partially

[43] This fact takes it's origin in Lie groups and is connected to Noether's theorem: Momentum is the generator of space translations and energy the generator of time translations.

[44] Just as momentum is the generator of space translations and energy the generator of time translation, charge can be understood as the generator of $\Lambda$ translations (for constant $\Lambda$). This idea is deeply linked with Kaluza and Klein's approach to think of $\Lambda$ as translations along a fifth "hidden" spatial dimension (see below).

discovered by the German physicist Walter Franz[45] as soon as 1939, before disappearing of sight during the second World War to be rediscovered ten years later by Ehrenberg and Siday (1949). The AB effect was rediscovered independently and popularized in the seminal paper of Aharonov and Bohm (1959) which had much more visibility and came another 10 years after. The effect was soon after observed by Chambers (1960), but the validity of the experimental setup was vividly questioned. Since then, the AB effect was detected multiple times in different context, gradually convincing the community of physicists of its significance (Tonomura et al. (1982), Timp et al. (1987), Geng et al. (2022)).

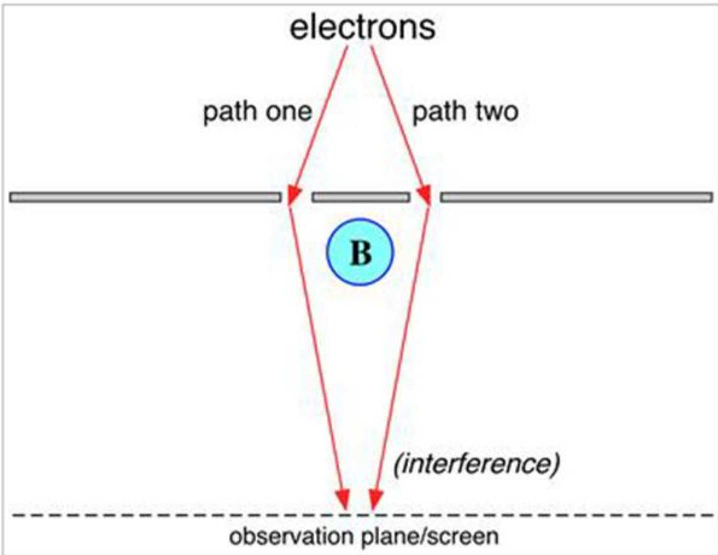



Figure 3: Schematic representation of The Aharonov-Bohm effect. (Credits: Wikimedia Commons).

As sketched on Fig. 3, the AB effect arises in the experimental context of electron interference. As in the usual double slit experiment, an electron beam is split in two paths which are then merged back together to produce an interference pattern. If we note $\psi_1$ and $\psi_2$ the values of the wavefunctions associated to the first and the second path respectively, the probability to observe an electron on the screen will be given by the value of $|\psi_1+\psi_2|^2$ on the screen. As a consequence of this, after the passage of multiple electrons, one would observe on the screen the characteristic oscillatory interference pattern.

Now, consider that a perfectly shielded solenoid is placed in between the two branches (in blue on the sketch). If current goes through the solenoid, a non-zero magnetic field $\vec{B}$ should exist within it but nowhere outside of it[46]. The electron beam and their associated wavefunctions cannot penetrate within the solenoid, such that $\psi$ is never in contact with $\vec{B}$.

Surprisingly, it is possible to prove (theoretically and experimentally) that the presence of this non-zero $\vec{B}$ within the solenoid will impact the shape of the interference pattern on the screen. Quantitatively, the interference pattern once current goes through the solenoid will be given by

$$\boxed{P(\tilde{x}) = |\psi_1 + e^{\mathrm{i}\Delta\varphi}\psi_2|^2 \propto \cos\left(\frac{\pi d}{\lambda l}\tilde{x} + \frac{\Delta\varphi}{2}\right)^2,} \tag{23}$$

where[47]

$$\Delta\varphi = -\frac{q}{\hbar}\Phi, \tag{24}$$

with $\Phi$ the flux of the magnetic field through the surface of the solenoid $\Sigma$:

$$\Phi = \iint_\Sigma \vec{B}\cdot\, \mathrm{d}\vec{S} = \oint_\gamma \vec{A}\cdot\, \mathrm{d}\vec{\ell}\,. \tag{25}$$

45 For a detailed account of the history of the AB effect, see Hiley (2013).

46 This is true in the limit in which the solenoid is infinitely long but can be realized in practice using shielding foil around the solenoid or toroidals ferromagnets.

47 The presence of the minus sign depends on a question of convention.

Here $\gamma$ denotes the closed loop formed by the combination of the first and the second path (we used Green-Ostrogradsky's theorem to transform the open surface integral of a curl into the closed integral of $\vec{A}$). As such, it seems that either $\vec{B}$ can act in a region in which it is zero, through some sort of non locality or $\vec{A}$, despite being a gauge variant quantity, must have some physical reality and act locally on $\psi$. This second view was the one proposed originally by Aharonov and Bohm (1959) in their original account of the effect and was greatly popularized by Richard Feynman in his famous lectures on physics (Feynman 1964). As we will further discuss later, explanations of the AB effect have been given in terms of space-time topology, according to which this effect is resulting from the manifestation of the space-time behaving as if it where not simply connected due to the presence of the solenoid (Hecht 2003).

## 6 Relativity and covariant electromagnetism

In Sec. 4, we saw how the gauge transformation appears as a redundancy in the mathematical formalism of the electromagnetic theory. In Sec. 5 however, we saw that gauge fields were unavoidable in the formalism of quantum theory and that they can act on particles wavefunctions in regions in which the electromagnetic fields are vanishing, hinting that, one way or another, gauge fields could have a physical meaning and could even be more fundamental than the force fields. We also saw, through the gauge argument, that imposing the requirements of phase invariance of the wave function seems to be at the origin of the wavefunction. We will now see that, in the context of special relativity, the gauge fields and the gauge invariance are not just a mathematical tool but are deeply rooted in the geometrical structure of space-time itself. The discovery of electromagnetic induction and its formulation within Maxwell theory revealed the profound links between the electric and magnetic fields. The advances of mathematical concepts at the end of the XIX$^{\text{th}}$ century, as well as the emergence of special relativity at the beginning of the XX$^{\text{th}}$ century, allowed a deeper realization that the electric and magnetic fields could be understood as two facets of the same entity, or more explicitly, as two different components of a single mathematical object. This idea was first formalized within the so-called electromagnetic tensor $F_{\mu\nu}$. Along with these investigations, it was also realized that the scalar and vector potentials, $V$ and $\vec{A}$, can identically be understood as two components of a unified object, the four-potential $A_\mu$. The four-potential $A_\mu$ plays a key role in the relativistic framework, as it allows to define $F_{\mu\nu}$ in a fully covariant fashion, without the need to separately manipulate the frame-dependent vectors $\vec{E}$ and $\vec{B}$. As for $F_{\mu\nu}$, profound geometrical meanings will later be attributed to $A^\mu$ within this relativistic context.

### 6.1 History: The birth of special relativity, an electromagnetic story

After the tremendous success of classical electromagnetism presented in Sec. 3, a major challenge that remained was to better understand the medium in which electromagnetic waves were supposed to propagate: the proposed luminiferous aether. These developments unfolded in parallel with the foundations of quantum mechanics, presented in the previous section. The famous experiment of Michelson and Morley (1887) yielded a null result for the detection of the 'aether wind', namely the expected signature of the motion of the Earth with respect to a stationary aether, which should have appeared in interferometric measurements as a directional difference in the propagation of light as the Earth travels around the Sun. This result did not by itself establish the full conceptual framework of special relativity, but it contributed decisively to the growing tension between classical electrodynamics and the mechanics of moving bodies. These tensions concerned notably the status of **inertial frames**[48], namely frames related, in the Newtonian setting, by translations at constant velocities.

This result gave rise to a variety of competing responses as the theoretical developments of FitzGerald (1889), Lorentz (1895) and Poincaré (1905b), who explored the transformations relating inertial frames in electrodynamics and the hypotheses, including length contraction, that could account for the

[48]Inertial frames already played a key role in classical mechanics, as frames in which Newton's laws are valid. More precisely, they are frames in uniform relative motion in which force-free bodies move rectilinearly and uniformly. Their definition and interpretation raise philosophical questions of their own (Maudlin 2012; Read 2023).

null result while preserving the general structure of Maxwellian theory. For a discussion of these devellopements, see Brown (2001) and Darrigol (2005). Such developments were not perceived immediately as a revolution. Michelson himself appears to have regarded the null result primarily as a failure of the original ether-drift programme — that is, a failure to detect something he still believed to exist — rather than as an immediate reason to abandon ether theories. Contemporary and later historical accounts also indicate that he was personally disappointed by the outcome and unconvinced by the more radical theoretical conclusions that would later be drawn from it (see e.g. Livingston (1979)).

The proposal of Einstein (1905b), leading to special relativity, did not merely infer a new kinematics directly from Michelson's result, but elevated the principle of relativity together with the constancy of the speed of light in all inertial frames to the status of fundamental postulates. The consequence was that the spatiotemporal relation between two events, as well as the notion of simultaneity, could no longer be regarded as absolute, but had to be understood as dependent on the state of motion of the observer. Hermann Minkowski (1908), who had been one of Einstein's professors, then provided the geometric reformulation that unified space and time into a single four-dimensional continuum, now known as spacetime. In Minkowski's framework, events are represented by spacetime coordinates $(ct, \mathbf{x})$, and the invariant interval between events replaces the Newtonian separation between absolute spatial distance and absolute temporal duration. This unification allowed a formulation of physics in which the distinction between space and time becomes frame-dependent. The challenge of a relativistic theory is to write equations in a covariant form, that is in such a way that their form is preserved under the so-called Lorentz transformations relating the coordinates of events as observed in different inertial frames, while leaving the value of the speed of light invariant. For discussions on the philosophical and geometric foundations of special relativity Maudlin (2012) and Read (2023).

### 6.1.1 Weber, Silberstein and the electromagnetic bivector

In 1901, Heinrich Martin Weber publishes the fourth editions of his lectures on partial differential equations of mathematical physics according to Riemann's lectures (Weber 1901). The lecture presents mathematical tools to solve complicated differential equations of physics, according to Riemann's own work. The fourth edition includes new original content from Weber, focused on classical electromagnetism. Weber notes in p.348 of the second volume that it is convenient to use the complex number $\mathfrak{E} + i\mathfrak{M}$, where $\mathfrak{E}$ and $\mathfrak{M}$ are the electric and magnetic fields respectively. With this new tool, two of the Maxwell equations without charges can be written as a single one as

$$\boxed{c\,\mathrm{curl}\mathfrak{E} + i\mathfrak{M} = i\frac{\partial(\mathfrak{E} + i\mathfrak{M})}{\partial t}} \tag{26}$$

where $c$ is the speed of light in vacuum[49]. This idea was further reused later by Silberstein in 1907 (Silberstein 1907a; Silberstein 1907b). Following the work of Hamilton (1866) on quaternions, Silberstein name bivectors the complex sum of two vectors $R_1$ and $R_2$ as $R_1 + iR_2$. This is the birth of a crucial concept in the context of what are known as Clifford algebra, which later will proove to be powerful to reformulate relativistic theories. Silberstein considers the "electromagnetic bivector" noted as $\eta = E_1 + E_2$ where $E_1$ and $E_2$ are the electric and magnetic fields respectively and rewrite following Weber the four Maxwell equations in vacuum as (Eq. I and II p.3 and 4)

$$\frac{\partial \eta}{\partial t} = -i\mathrm{curl}\eta; \tag{27}$$

$$\mathrm{div}\eta = 0. \tag{28}$$

Considering the complex conjugate $\eta' = E_1 - iE_2$ and noting the dot and the cross product between two vectors $R_1$ and $R_2$ respectively as $R_1R_2$ and $VR_1R_2$, Silberstein continues by studying interesting properties of $\eta$ as its use to compute the electromagnetic energy density as $\eta\eta'/2$ and the Poynting vector as $iV\eta\eta'$.

[49]the factor $c$ simply translates the choice of units of Weber.

The quantity $\widetilde{F} = \vec{E} + \mathbb{i}\vec{B}$ (in modern notations) is named the Riemann-Silberstein or Weber vector. Besides its applications to simplify equations in classical electromagnetism, this vector was of peculiar interest for its possible interpretation as a wavefunction for the photon (Bialynicki-Birula and Bialynicka-Birula 2013; Kiessling and Tahvildar-Zadeh 2018).

### 6.1.2 Minkowski and after: modern formulation

While both $\vec{E}$ and $\vec{B}$ are unified in the Weber vector, the way this vector transform under Lorentz transformation is unclear and the equations written with it are not explicitly covariant. Put more simply: while $\widetilde{F}$ provides a unification of the electromagnetic fields in a single object, it is unclear how this object would fit in a relativistic theory.

A better candidate was proposed by Hermann Minkowski (1908) in his unified formulation of relativity within space-time. Noting $\mathfrak{e} = \varepsilon\mathfrak{E}$ and $\mathfrak{M} = \mu\mathfrak{m}$ where $\mathfrak{E}$ and $\mathfrak{M}$ are the electric and magnetic field and $\mu$ and $\varepsilon$ are the vacuum permeability and permittivity constants. Minkowski becomes able to reveal a new hidden symmetry of electromagnetism contained within electromagnetism under a special relativistic context. Writing (p.367) $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = it$ and writing $s_1$, $s_2$, $s_3$, $s_4$ for $\mathfrak{s}_x$, $\mathfrak{s}_y$, $\mathfrak{s}_z$, $i\varrho$ where $\mathfrak{s}_i$ are the components of the current density (written $\vec{j}$ in modern notations) and $\varrho$ is the electric charge density (noted $\rho$ in modern notations). Considering $4 \times 4$ matrices $f$ and $F$ which components are (p.368): $f_{23}$, $f_{31}$, $f_{12}$, $f_{14}$, $f_{24}$, $f_{34}$ $\mathfrak{m}_x$, $\mathfrak{m}_y$, $\mathfrak{m}_z$, $-i\mathfrak{e}_x$, $-i\mathfrak{e}_y$, $-i\mathfrak{e}_z$ and also $F_{23}$, $F_{31}$, $F_{12}$, $F_{14}$, $F_{24}$, $F_{34}$ $\mathfrak{M}_x$, $\mathfrak{M}_y$, $\mathfrak{M}_z$, $-i\mathfrak{E}_x$, $-i\mathfrak{E}_y$, $-i\mathfrak{E}_z$; $f, F$ are anti-symmetric as $f_{kh} = -f_{hk}$ and $F_{kh} = -F_{hk}$. Then (Eq.A p.368) gives the two charged Maxwell equations while (Eq.B p.368) are the two charge-free Maxwell equations. They are both reproduced below:

$$\begin{aligned}
\frac{\partial f_{12}}{\partial x_2} + \frac{\partial f_{13}}{\partial x_3} + \frac{\partial f_{14}}{\partial x_4} &= s_1 \\
\frac{\partial f_{21}}{\partial x_1} + \frac{\partial f_{23}}{\partial x_3} + \frac{\partial f_{24}}{\partial x_4} &= s_2 \\
\frac{\partial f_{31}}{\partial x_1} + \frac{\partial f_{32}}{\partial x_2} + \frac{\partial f_{34}}{\partial x_4} &= s_3 \\
\frac{\partial f_{41}}{\partial x_1} + \frac{\partial f_{42}}{\partial x_2} + \frac{\partial f_{43}}{\partial x_3} &= s_4
\end{aligned}$$

$$\begin{aligned}
\frac{\partial F_{34}}{\partial x_2} + \frac{\partial F_{42}}{\partial x_3} + \frac{\partial F_{23}}{\partial x_4} &= 0 \\
\frac{\partial F_{43}}{\partial x_1} + \frac{\partial F_{14}}{\partial x_3} + \frac{\partial F_{31}}{\partial x_4} &= 0 \\
\frac{\partial F_{24}}{\partial x_1} + \frac{\partial F_{41}}{\partial x_2} + \frac{\partial F_{12}}{\partial x_4} &= 0 \\
\frac{\partial F_{32}}{\partial x_1} + \frac{\partial F_{13}}{\partial x_2} + \frac{\partial F_{21}}{\partial x_3} &= 0
\end{aligned}$$

The letters $f, F$ are chosen for the word "Feld" (field in German) as "s" is for the Strom (meaning current). The introduction of the electromagnetic fields in a single covariant object $F$ was refined in other follow-up working treating of relativstic dynamics, but the main conceptual shift was already in place (Abraham 1910; Bateman 1910; Born 1909; Conway 1911; Einstein and Grossmann 1913; Ignatowski 1910; Kottler 1912; Laue 1911; Lewis and Wilson 1912; Silberstein 1912; Sommerfeld 1910). [50].

### 6.1.3 The single covariant potential: grouping $V$ and $\vec{A}$

As discussed in Sec. 2, following the work of Maupertuis, Euler and Lagrange, it was found that the equations of mechanics could be recovered by minimizing the action. An action is the integral over time of a Lagrangian, $L = T - U$ where $T$ is the kinetic energy and $U$ is the potential energy that we already encountered. We do not give to much conceputal details on actions, but it is an extremely important concept for our discussions for the following reasons:

- All the known laws of physics known up to this days are believed to be expressed at the most fundamental level by a Lagrangian. Lagragians make explicit the symmetries of a theory, as gauge invariance. **Because $L$ is expressed in terms of energies, it uses the potentials explicitly as a fundamental quantities to describe interactions, not the force fields themselves.**

[50]For a discussion of these sources, see the entry: https://en.wikiversity.org/wiki/History_of_Topics_in_Special_Relativity/Electromagnetic_tensor

- From the lagrangian $T - U$ one can obtain the total energy function, known as the Hamiltonian $H = T + U$ (using what is known as a canonical transformation). In quantum mechanics, it is standard to derive the Schrödinger equation (Eq. 17) from the Hamiltonian $H$ using a procedure known as the "correspondence principle". As such, **the historical and conceptual transition from a classical to a quantum theory** relies heavily on $L$ and $H$.

We will see in the next section how this second point played a key role in the introduction of the potentials in a quantum context. Simply in order to compare with the historical expressions below, we note that the Lagrangian allowing to recover the Lorentz force is, in modern notations $L_{\text{Lorentz}} = m\vec{v}^2/2 + q\vec{v}\cdot\vec{A} - qV$, while the one associated to the Maxwell equations is $L_{\text{Maxwell}} = \varepsilon_0\vec{E}^2/2 - \vec{B}^2/(2\mu_0)$.

After the developments of Maxwell's theory of electromagnetism, attempts were proposed to derive classical electromagnetism from an action principle, as most of classical mechanics was following Lagrange's theory. In 1900, Joseph Larmor was able to recover both the Lorentz force and Maxwell equations using a least Action principle build in analogy with the theory of elastic solids applied to aether (Larmor (1900), Chapter VI). His derivations are rather technical and hard to follow such that we will not detail them here. The same results were derived independently in a strikingly modern fashion by Karl Schwarzschild in 1903. The author starts by writing the Lorentz force in terms of the electric potential $\Phi$ and the vector potential noted $\Gamma_x, \Gamma_y, \Gamma_z$ and further shows that the force can be recovered by considering the action $\int \, \mathrm{d}t(-T + \sum eL)$, where $T$ is the kinetic energy, $e$ the charges and

$$\boxed{L = \Phi - v_x\Gamma_x - v_y\Gamma_y - v_z\Gamma_z} \tag{29}$$

that Schwarzschild calls the "elektrokinetisches Potential" (p.127), hence understanding that both electric and magnetic forces can be recovered in a general fashion by considering the scalar potential given by $V - \vec{v}\cdot\vec{A}$ or $u_\mu A^\mu$ in modern notations. The potentials are the key to formulate this action (and the interaction term $L$ cannot be written using the fields themselves). This is also the first indication that the quantities $\Phi/V$ and $\Gamma_i/A_i$ could be treated on equal footing in a relativistic context, even though, of course, this was not concievable at the time of Schwarzschild's derivation.

Schwarzschild further proceeds by showing that the full electromagnetic theory can be obtained by adding to the kinetic action the quantity (p.128)

$$\boxed{\int \, \mathrm{d}t \; \mathrm{d}\omega \left\{ \frac{H^2 - K^2}{8\pi} + \chi L \right\}} \tag{30}$$

where $\omega$ is a volume element, $H$ and $K$ are respectively the magnetic and electric fields and $\chi$ is the charge density. This is thus the full Lagrangian density for electromagnetism including both the Lorentz force and Maxwell equations, which will remain in this exact form up to this day. Schwarzschild's work is arguably visionary in deriving so simply and clearly these results two years before Einstein's special relativity. However, no mention is made by Schwarzschild of how gauge invariance manifests itself at the Lagrangian level. This discussion will not appear in printed works until much later in Landau and Lifshitz (1941). On this point see Bergmann (1946) and Jackson and Okun (2001). As such, the potentials appeared early as crucial entities at the Lagrangian level.

The next conceptual revolution was to realise that $V$ and $\vec{A}$ are to be understood as components of a unique four dimensional object. It is known that Henri Poincaré anticipated some of the mathematical structure underlying what would become special relativity. In a series of notes and papers culminating in his memoir *Sur la dynamique de l'électron* (Poincaré 1905a; Poincaré 1906), he demonstrated that the transformations leaving Maxwell's equations invariant form a group — the Lorentz group — and emphasized that these transformations apply to space and time coordinates rather than serving merely as computational devices (For more details on Poincaré's contribution, see also Darrigol (2000) and Miller (1973)). For our interest now, Poincaré showed that the scalar potential $\psi$ and the vector potential with components noted $(F, G, H)$ (following Maxwell) transform under Lorentz transformations in such a way that Maxwell's equations retain their form (Miller 1973; Poincaré 1906). Although he did not formulate these quantities using the later four dimensional formalism, the transformation laws he derived are suggestive of a possible unification of $\psi$ and $F, G, H$, as $t$ and $x, y, z$.

Immediately after, Roberto Marcolongo (1906) reformulated Maxwell's equations in an explicitly four-dimensional setting. While the full aparatus of space-time geometry was not yet in its final form, the electromagnetic quantities — including the potentials — were organized within a four-dimensional framework. This represents one of the earliest systematic attempts to cast electrodynamics into a genuinely four-dimensional formalism.

As for the electromagnetic tensor, the complete four dimensional formulation of the potential was done by Hermann Minkowski in his full reformulation of special relativity in a four dimensional geometric context. In a leture held in 1907 and only published later in 1915, Minkowski clearly introduces $\psi_1, \psi_2, \psi_3, \psi_4$, with $\psi_4 = i\Phi$ the electric potential (Minkowski 1915).

He further introduced what will later be identified as the Faraday tensor, here dubbed "Traktor":

$$\boxed{\psi_{jk} = \frac{\partial \psi_k}{\partial x_j} - \frac{\partial \psi_j}{\partial x_k}} \tag{31}$$

and interpreted the components of this object to be the electromagnetic fields. This work anticipated the full reformulation of Maxwell equations using $F_{\mu\nu}$ described in the previous section.

## 6.2 Concepts: covariant electromagnetism

As we will now see, the framing of relativity will allow for a deep unification of electromagnetic phenomenon, in which space and time are fused and the gauge transformations take a natural place.

In special relativity, time is considered as an extra dimension, and usual vectors are replaced by four component objects, the four vectors. Change of frames between two frames moving relative to one another at constant velocity (**inertial frames**) are understood as rotations in space-time and will mix the time and space components of the 4-vectors. Such transformations are called **boosts**. In the following, we chose natural units, in which the speed of light is set to unity ($c = 1$). First, we will introduce the potential 4-vector $A^\mu = (V, \vec{A})$ as well as the current four-vector $J^\mu = (\rho, \vec{j})$. These two quantities vectors quantities merge in a single object the notions related to gauge fields and charge densities/currents respectively.$\mu$ is the coordinate index, which can take values $0, 1, 2, 3$. Note that, in the relativistic context, energy $E$ and momentum $\vec{p}$ also fuse to become the 4-impulsion $p^\mu = (E, \vec{p})$. As such, looking at $A^\mu$, we are better able to understand why, as we saw previously in a quantum context, $V$, the temporal part of $A^\mu$ is associated with energy and $\vec{p}$, its spatial part, to momentum.

The covariant object containing the electromagnetic fields is the so-called **Faraday** or **Maxwell tensor** $F = F_{\mu\nu}\, \mathrm{d}x^\mu \otimes\, \mathrm{d}x^\nu$ which components are defined as[51]

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu} \tag{32}$$

Here both $\mu$ and $\nu$ are indices which can take values between 0 and 3, such that $F_{\mu\nu}$ is a 16 component objects. It is not a simple matrix however, but the components of a geometrical object $F$ known as a tensor, which transforms appropriately under Lorentz transformations. Using the definition of $\vec{E}$ and $\vec{B}$ in terms of $\vec{A}$ and $V$ (Eq. 13), we can express the components of $F$ in a given inertial frame $(t, x, y, z)$ as

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \tag{33}$$

Why would we bother using this instead of $\vec{E}$ and $\vec{B}$? As $F$ is a tensor, equations in which it appears are so-called covariant and will remain true in any frames. As such, relations involving $F$ capture physical laws.

[51]From now on, $\partial_\mu$ will be used as a shorthand notation for the derivative $\frac{\partial}{\partial x_\mu}$.

First, from the definition of $F$, it is possible to find

$$\boxed{\partial_\sigma F^{\mu\nu} + \partial_\mu F^{\nu\sigma} + \partial_\nu F^{\sigma\mu} = 0.} \tag{34}$$

Now, with a bit of work replacing the components of $F$ using Eq. 33, it is possible to show that the above equation is nothing else than the two charge-free Maxwell equations. This is the second equation found by Minkowski in his 1908 paper. The charged Maxwell equations become (they can be found through a action minimisation principle which we will not discuss here):

$$\boxed{\partial^\mu F_{\mu\nu} = J_\nu,} \tag{35}$$

which correspond to the two charged Maxwell equations and is the modern form of the first equation found by Minkowski in his 1908 paper. We assume here and in the rest of the text that when an index is repeated up and down (here $\mu$) it is summed over (Einstein summation convention). Let us stress again that these equations, expressed with tensors, are manifestly covariant i.e. they remain valid in any inertial frame. As they are perfectly equivalent to Maxwell equations, we demonstrated that relativity was already encompassed, but hidden, within classical electromagnetism. The continuity equation also follows from the definition of $F$ in combination of Maxwell equations and is written $\partial_\mu J^\mu = 0$. The Lorentz force (Eq. 5) becomes also fully expressed in terms of the unified $F_{\mu\nu}$ object as:

$$\boxed{a_\mu = \frac{q}{m} F_{\mu\nu} u^\nu,} \tag{36}$$

where $a_\mu$ and $u^\nu$ are the four dimensional acceleration and velocity. The covariant formulation of electromagnetism conceals a profound elegance beneath its formal structure. In this framework, the electric and magnetic fields emerge as two aspects of a single entity—the electromagnetic field tensor—and they transform into one another under changes of inertial reference frame. In this sense, the magnetic field is not an independent entity, but can be understood as an electric field seen in a moving frame.

Let us now look at gauge invariance. The two gauge transformations of Eq. 14 can be contained in a single covariant expression as

$$\boxed{A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x).} \tag{37}$$

We see that the two gauge fields $\vec{A}$ and $V$ are treated on the same footing in this framework, as respectively the space and time components of the 4-vector $A$. As such, the strange difference of behaviour between $\vec{A}$ and $V$ in Eq. 14 appears natural in four dimensions. As the partial derivatives commute, it is easy to show that $F$ remains unchanged under such a transformation of the gauge fields:

$$F'_{\mu\nu} = \partial_\mu A_\nu + \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \partial_\nu \partial_\mu \Lambda = F_{\mu\nu}. \tag{38}$$

# 7 Relativistic fields and gauge potentials

## 7.1 History: Relativistic fields and gauge potentials, some geometrical origins

The decade between 1918 and 1929 saw the reunion of quantum mechanics and special relativity, context in which gauge theories emerged in their modern form. The introduction of the gauge potential in a quantum context was done using several approaches proposed independently in multiple parallel ways, based on different motivations:

- **The Relativistic path (Fock, Gordon, Klein):** motivated by the incompatibility of the Schrödinger equation with relativity (Eq. 17), authors were seeking a for a relativisitic wave equation including electromagnetic interactions. A first relativistic equation was proposed by Fock (1926a) and Fock (1926b) and Gordon (1926). This equation was not conceptually satisfying and lead to the proposition of his relativistic equation by Dirac (1928) .

- **The Geometric path (Weyl's Weltgeometrie):** Hermann Weyl (1918a) attempted to unify gravity and electromagnetism by generalizing the geometry of general relativity. Although his original theory was empirically refuted, it provided the mathematical template for the introduction of gauge invariance in the quantum context. The first steps in that directions were proposed by Schrödinger (1922) and London (1927) and achieved by Weyl (1929b) himself in the context of the equation proposed by Dirac (1928).
- **The higher-dimensional path (Kaluza-Klein):** Theodor Kaluza (1921) and Oskar Klein (1926a) showed that the vector potential could be interpreted as the metric of a five dimensional space-time. This approach hinted at a purely geometric origin for the vector potential, where "gauge transformations" correspond to rotations in a hidden fifth dimension. This theory revealed to be empirically inadequate but will strongly inspire later developments as string theory.

As such, the insertion of $\vec{A}$ in a quantum theory was thus not done "for itself", but always in a broader context within the emultion generated by special and general relativity. A commented and translated recollection of some of the key works of this period can be found in O'Raifeartaigh (1997). A striking historical fact emerging from the above list is that several foundational advances in the modern understanding of gauge theories emerged from early attempts to unify gravity and electromagnetism within a single geometric framework: Weyl's Weltgeometrie and the Kaluza-Klein theory. Although these proposals were ultimately empirically inadequate in their original form, they were far from sterile. On the contrary, they introduced structural ideas that became central to modern gauge theory and their conceptual legacy continues to shape active research programs, which are their direct successors as conformal field theories, Weyl cosmology and string theory.

### 7.2 The relativistic path: seeking a relativistic generalization of the Schrödinger equation

In the months following Schrödinger's 1926 publication of non-relativistic wave mechanics, several physicists—most notably Vladimir Fock, Oskar Klein, and Walter Gordon—attempted to reconcile the theory with special relativity. The challenge was that the Schrödinger equation was first-order in time and second-order in space, violating the Lorentz-covariant requirement that space and time be treated on equal footing. In his June 1926 paper, Vladimir Fock (1926a) sought an equation that would remain invariant under both coordinate transformations and electromagnetic gauge transformations. His starting point was the Lagrangian for a charged relativistic particle, which generalizes in modern notations what was found by Schwarzschild and refined by Minkowski and others in a relativistic context. Deriving the Hamiltonian[52] associated to this Lagrangian and using the correspondence principle, Fock could find a relativistic version of the Schrödinger equation and identify the minimal coupling rule to go from a free to a coupled equation. He recognized that for the equation to hold, the "action" (the phase $p$) must change in tandem with the potentials. As noted in his original paper (Fock 1926a), the transformation laws were expressed as (Eq. 5 p.227):

$$\boxed{\left.\begin{aligned} \mathfrak{A} &= \mathfrak{A}_1 + \operatorname{grad} f \\ \varphi &= \varphi_1 - \frac{1}{c}\frac{\partial f}{\partial t} \\ p &= p_1 - \frac{e}{c} f \end{aligned}\right\}}$$

This set of equations clearly state for the first time the complete gauge transformation of the potentials along with the wavefunction in a quantum context. However, it was quickly realised by Gordon (1926), in his study of the Compton effect, and Klein (1926a) that the relativistic generalization of the Schrödinger equation proposed lead to negative probability densities and could thus not be interpreted correctly in a quantum context.

[52] More accurately: Fock used the Hamilton-Jacobi formalism. It seems that this formalism played a critical role in the early considerations about wave-mechanics as it makes more explicit the proximity with optics, and more precisely the limit of light waves behaving like rays through the eikonal approximation (It notably allowed Bohm later to derive his pilot wave theory).

The resolution of these issues required a different strategy. Paul Dirac (1928) constructed a relativistic wave equation linear in both time and space derivatives (Eq. 39), thereby ensuring Lorentz covariance while maintaining a positive-definite probability density. Dirac's equation not only resolved the interpretational problems of the Klein–Gordon theory but also predicted the existence of antiparticles, inaugurating the modern relativistic theory of spin-1/2 particles.

### 7.3 The geometric path: Weyl's Weltgeometrie

It is difficult to overstate the role that Hermann Weyl played in the formulation of gauge theories in their modern form. Weyl was the first to anticipate the underlying geometrical structure of gauge theories, which will be at the heart of Part III. Weyl's developments, and their subsequent influence in physics, occurred in parallel with major transformations in mathematics. These were driven both by attempts to reconcile Bernard Riemann (1854)'s differential geometry — centred on manifolds and local structures — with Felix Klein (1872)'s Erlangen program, which conceived geometry in terms of symmetry transformations, and by the conceptual demands imposed by general relativity. From the mathematical side, this gradually lead to the realisation that the abstract spaces of Klein's geometry could be located at each point of a manifold in the sense of Riemann. This realisation was done gradually by Cartan (1922) and Cartan (1926) in attempts to refine and go beyond the formalism of general relativity. This odyssey took a mature form in the work of his student, Ehresmann (1951), giving birth to what is now known as the fiber bundle formalism (Appendix D.2). The full realization that gauge theories could be expressed naturally in the language of fiber bundles emerged only much later; to our knowledge, the first explicit formulations appear in Trautman (1970) and reached conceptual maturity with Wu and Yang (1975a), after the formal developement of modern gauge theories. For further discussion, see our contribution on this topic[53].

Einstein's general relativity interprets gravitation as a manifestation of the geometry of space–time (Einstein 1915). In curved geometry, it is impossible to compare directly the direction of vectors located at different points of space–time. Such a comparison requires transporting one vector to the location of the other through a procedure known as **parallel transport**. In a curved space–time, the outcome of parallel transport depends on the path followed. The mathematical structure that defines how vectors are transported is called a **connection**. We will return to these notions in Sec. 13.1. In general relativity, the connection preserves lengths and angles.

In his mathematical work, Weyl (1918b) isolated the notion of connection ("Zusammenhang", literally "holding together") as an independent geometric structure, emphasizing that it need not be uniquely determined by the metric. Shortly thereafter, in his celebrated 1918 paper Weyl (1918a), Weyl proposed a radical generalization. He considered it conceptually unsatisfactory that only the orientation of vectors required comparison rules between neighbouring points, while their lengths remained invariant. Weyl therefore introduced additional terms to the connection, allowing lengths themselves to vary under transport.

In Weyl's geometry, the value of length at a point becomes intrinsically meaningless and may be locally rescaled through what is now called a conformal transformation[54]. Weyl observed that, under such a local rescaling, the part of the connection transporting lengths must transform in precisely the same way as the electromagnetic potential of electromagnetism in order for the equations of motion to remain invariant. This invariance under local changes of scale — initially called "Maßstabinvarianz" (scale invariance) and later "Eichinvarianz" (gauge invariance) following discussions with Einstein — introduced the terminology that survives today (Scholz 2004). Weyl thus attempted a unification of gravitation and electromagnetism by identifying the electromagnetic potential with the connection associated with local changes of scale.

In some sense, this proposal represented a prototype of the gauge principle: a physical interaction emerging from the requirement of local symmetry invariance. Although mathematically profound, Weyl's theory faced an immediate physical objection. While Einstein described Weyl's theory as "a coup of genius of the first rate...", he had to admit that "...although your idea is so beautiful, I have

[53] https://leovacher.github.io/files/connexion-Vacher-en.pdf

[54] A conformal rescaling is a transformation of the form $g \to e^{\Lambda(x)} g$ (in modern notations), where $\Lambda(x)$ is a scalar function. It changes lengths while preserving angles.

to declare frankly that, in my opinion, it is impossible that the theory corresponds to nature"[55]. He in fact pointed out that if lengths depended on the path taken during transport, then the size of atoms — and consequently the frequencies of their spectral lines — would depend on their past history. The empirical existence of sharp spectral lines therefore ruled out Weyl's original theory.

Despite the failure of Weyl's original proposal, his central insight survived. The decisive turning point arrived with the development of quantum mechanics. An important step was performed by Schrödinger (1922). In this work, Schrödinger starts from Weyl's 1918 proposal and try to look at its application in the newly emerging quantum context, already indicating that, despite its empirical inadequacy, such proposal received much attention. Schrödinger points that, in Weyl's formalism, the length of a vector is not absolute but changes during transport according to $\mathrm{d}l = -l\varphi_i \, \mathrm{d}x^i$, where $\varphi_i$ are the components of Weyl's length connection. As such, after motion along a path the length is multiplied by an exponential factor, called the "Streckenfaktor" $\exp\big(-\int \varphi_i \, \mathrm{d}x^i\big)$. Schrödinger then adopts Weyl's identification of $\varphi_i$ with the electromagnetic potentials (up to a universal constant) and notices that the exponent becomes an integral involving the scalar and vector potentials along the electron's trajectory — mathematically the same structure as the electromagnetic contribution to the classical action in Hamilton–Jacobi theory. Schrödinger then considers closed path around loops. The reason is that, at the time, the major challenge of quantum theory is to explain the Bohr–Sommerfeld conditions – required to find the quantized orbits of atoms – stating that over periodic motions satisfy $\oint p \, \mathrm{d}q = nh$. Schrödinger demonstrates, through detailed analyses of Kepler motion, the Zeeman effect, and the Stark effect, that these quantum conditions force the exponent of Weyl's transport factor to be an integer multiple of Planck's constant for every quasi-periodic orbit. Thus, although Weyl's geometry predicts a path-dependent change, quantized motions make the accumulated factor effectively periodic: after one quantum cycle the transported quantity almost reproduces itself. This exponential thus behaves exactly like what we now call a phase factor, even though wavefunctions have not yet been conceptualized. The quantity whose exponential appears is the classical action $S$; three years before Schrödinger's 1926 wave equation, the action already played the role of a phase-like variable because only its value modulo $h$ mattered in quantization. Schrödinger himself recognized a tension here: if the exponential is real, it changes physical lengths and contradicts atomic spectroscopy (Einstein's objection to Weyl), but if the proportionality constant is chosen to be imaginary, the exponential becomes $e^{iS/\hbar}$, a pure rotation of unit modulus that leaves magnitudes unchanged. The appearance of the imaginary unit is therefore not a mathematical trick but a physical necessity: it converts Weyl's unacceptable real scale change into an unobservable change of phase. What Schrödinger uncovered, therefore, was not yet gauge invariance but the deeper structural fact that electromagnetism naturally enters physics through exponentials of line integrals of the potential whose physically meaningful content lies in their values around closed paths. Let us stress again that such a work was done three years before the formulation of wave mechanics. Nevertheless, although Schrödinger sensed that a profound idea was involved, he did not yet know how to interpret it, referring instead to a "remarkable property of the electron," as reflected in the title of his paper. While Schrödinger (1926a) does not cite his previous work, letters he sent to Einstein and Lande reveal that these reflexions inspired him towards his formulation of wave mechanics. A discussion of this topic as well as the realisation that iamginary numbers where key in a quantum context, can be found in Yang (1987).

Schrödinger's work attracted the attention of Fritz London, who argued that his 1922 ideas should be extended within the framework of the newly developed wave mechanics and expressed surprise that Schrödinger himself had not already taken this step (see the letter reproduced at the end of Yang (1987)). London (1927) represents the decisive conceptual turning point that transforms Weyl's geometrical idea from an unsuccessful theory of variable physical lengths into the precursor of modern gauge theory by identifying the transported quantity not with a material scale but with the phase of matter itself. He proposes a radical reinterpretation of Weyl's theory: the length connection should not act on physical vectors but on the phase of the de Broglie wave or on Schrödinger's matter wave. London states explicitly that "der physikalische Gegenstand ist gefunden … die komplexe Amplitude der de Broglieschen Welle"[56] ("the physical object has been found: the complex amplitude of the

[55]See Straumann (1996) and references therein.

[56]London (1927) p.380

de Broglie wave"), which thus becomes the prototype of Weyl's gauge quantity[57]. At this moment the appearance of the imaginary unit ceases to be optional: to match the oscillatory character of matter, the previously undetermined proportionality factor in Weyl's transport law must contain $i/h$, turning Weyl's real exponential rescaling into a complex phase factor of unit modulus. Einstein's objection disappears automatically: phases may vary path-dependently without affecting measurable magnitudes. This represents the essential conceptual inversion transforming Weyl's failed classical theory into the precursor of modern gauge theory.

Weyl fully embraced this reinterpretation in the short paper (Weyl 1929b) and the longer follow-up Weyl (1929a), in which he reformulated gauge invariance as a symmetry under local phase transformations of quantum matter fields. In these works, Weyl introduces Dirac equation within general relativity (using the vierbein formalism). These papers lay explicitly all elements of the modern formulation of gauge theories[58]. However, Weyl realises still that $\varphi_p$ plays a role analoguous to the length connection of his 1918 theory. He states explicitly for the first time the **minimal coupling rule**: "In the Dirac theory the influence of an electromagnetic field is taken into account by replacing the operator $\frac{\partial}{\partial x_p}$, affecting the $\psi$ by $\frac{\partial}{\partial x_p} + i\varphi_p$." (p.329 Weyl (1929b)). This is in turns leads Weyl to give the first explicit formulation of his **gauge principle**, stated explicitely in the introduction of his second paper: "it seems to me that this new principle of gauge invariance, which follows not from speculation but from experiment, tells us that the electromagnetic field is a necessary accompanying phenomenon, not of gravitation, but of the material wave-field represented by $\psi$" (p.122 of O'Raifeartaigh (1997)). The motivation for the gauge principle is a principle of locality, analogous to the invariance of spinors under local Lorentz transformations: "I prefer not to believe in distant-parallelism for a number of reasons. First my mathematical intuition objects to accepting such an artificial geometry; I find it difficult to understand the force that would keep the the local tetrads at different points and in rotated positions in a rigid relationship." (p.122 of O'Raifeartaigh (1997)). Using Noether (1918) theorem, as he already attempted to in 1918, he further showed that invariance under global phase transformations implies the conservation of electric charge, reinforcing the deep connection between symmetry and conservation laws. This insight marked a decisive conceptual shift: gauge symmetry was no longer a mathematical artifact but a principle constraining physical laws themselves (see Brading (2002)). Almost simultaneously, Fock (1929) independently derived the same structure from quantum mechanical considerations, further consolidating the modern understanding of electromagnetic interactions as arising from local symmetry requirements. From this point onward, gauge invariance ceased to be associated with scale transformations and became fundamentally linked to internal phase symmetry — a reinterpretation that laid the conceptual foundations for modern particle physics.

Historical analyses of Weyl's contribution and philosophy can be found in (Eckes 2014; Scholz 2018; Scholz 2020; Scholz 2022), while conceptual discussions are provided in Straub (2005).

### 7.4 The higher-dimensional path: Kaluza-Klein theories

While Weyl sought to unify fields by changing the geometry of four-dimensional spacetime, Theodor Kaluza (1921) sought to unify them by adding an additional fifth dimension. That is: rather than modifying the notion of length comparison, Kaluza pursued unification by enlarging the geometric arena itself. In his 1921 paper, he suggested that the apparent coexistence of gravitational and electromagnetic interactions might reflect an incomplete description of spacetime, whose true structure possesses an additional coordinate. In his introduction, Kaluza presents his own work as an alternative route toward Weyl's 1918 unificatory goal[59]. His starting point can be understood in modern terms as a formal analogy between the curl-like expression of the electromagnetic field strength $F$ in

[57] A modern reinterpretation is as follows: the conformal rescaling invariance of the metric proposed by Weyl $g \to e^{\Lambda} g$, should instead be replaced by an imaginary phase on the quantum matter field $\psi \to \psi e^{\frac{i}{\hbar}\Lambda}\psi$.

[58] In these works, Weyl introduced the spin-connection, allowing for a conceptual path "out of the tangent bundle". However, he did not interpret $\varphi_p$ as a connection, in the technical sense, as was the length connection in his original 1918 theory. At its early stages, Weyl was even quite dubious of to the geometrical developments proposed by Cartan whish would allow such a conclusion. For more, see our own contribution linked above as well as Bell and Korté (2016).

[59] Kaluza was indeed inspired by Weyl's original work, and contacted Einstein – who responded very positively – as soon as 1919 (see O'Raifeartaigh and Straumann (1998b)).

terms of derivatives of the four-potential $A$, and the expression of the Christoffel symbols in terms of derivatives of the metric $g$. This observation led Kaluza to conclude that "one is led almost inevitably to an initially unattractive conclusion," namely to propose the "rather strange idea"[60] that a fifth dimension must be added to spacetime. Kaluza proposes to set $g_{0k} = 2\alpha q_k$ and $g_{00} = 2h$, where 0 is the index of the fifth dimension $x^0$, $q_k$ is the four-potential, and $h$ is an additional scalar quantity in the model which "manifests itself here essentially as a negative gravitational potential," while $\alpha$ is a constant later related appropriately to the gravitational constant. Kaluza further adds a so-called cylindrical condition, namely that derivatives along the fifth coordinate should vanish, thus permitting only four-dimensional variations. In this framework, certain components of the Christoffel symbols involving the fifth dimension become proportional to the electromagnetic field strength $F$, although Kaluza also notes the appearance of additional unwanted terms at an intermediate stage. Furthermore, after computing the curvature and under an explicit weak-field approximation, one finds that the higher-dimensional Einstein equations decompose into three sets of equations: the four-dimensional Einstein equations, Maxwell's equations, and an additional Poisson equation for the scalar $h$. In this way, electromagnetism appears not as an independent interaction but as part of spacetime geometry itself. Under further approximations, the Lorentz force law also arises from geodesic motion in five dimensions: the motion of a charged particle in electromagnetic and gravitational fields corresponds to free geodesic motion in the enlarged space. Furthermore, Kaluza identifies that the electric charge appears to be linked to the fifth component of momentum. While Kaluza is fully aware of the limitations of his formulation, he concludes that "it is hard to believe that the derived relationships [...] represent nothing more than a malicious coincidence".

Oskar Klein (1926a) recasts Kaluza's five-dimensional theory in the setting of the new wave mechanics of Schrödinger. Klein assumes a periodic, more precisely harmonic, dependence on the fifth coordinate. He is able to write down a five-dimensional wave equation which, in suitable limits and in particular cases, behaves like a version of the Schrödinger equation coupled to electromagnetism. Importantly for the later history of gauge theories, Klein notes that if a coordinate transformation involving the fifth dimension is performed, "a gradient of a scalar has to be added" to the quantities identified with the vector potential; in modern terms, this is naturally interpreted as a precursor of a gauge transformation, even though Klein does not formulate an autonomous gauge principle in the later sense. Towards the end of the paper, Klein adopts an Ansatz which, from a modern point of view, suggests that the effect of such a transformation of the fifth coordinate would be associated with a phase factor of the wave field, even though Klein does not explicitly discuss it in those terms. In a later note, Klein (1926b) further interprets the quantization of electric charge as a quantum consequence of the periodicity of the fifth dimension.

Despite its conceptual elegance, the Kaluza–Klein proposal lacked immediate empirical adequacy. The introduction of an unobservable fifth dimension and the auxiliary scalar field had no experimental support, and the compactification scale required by Klein's quantum interpretation lay far beyond accessible energies. As a result, the theory remained largely speculative and attracted limited attention for several decades. Interest revived only after the development of non-Abelian gauge theories in the 1950s, when it became clear that internal symmetries could be interpreted geometrically in higher dimensions (Kerner 1968). However, it was quickly realised that standard formulations of Kaluza-Klein theories lead to predictions in clear contradictions with experiments (For a review see Wesson (1999)). From the 1970s onward, Kaluza–Klein ideas reappeared in unified field theories, supergravity, and later string theory, where extra dimensions became a central structural ingredient rather than an ad-hoc hypothesis. Although high empirical constraints have been put on the possible existence of extra dimensions (e.g. CMS Collaboration et al. (2011)), the Kaluza–Klein framework has persisted as a guiding paradigm, shaping modern approaches to unification and providing one of the earliest geometric interpretations of gauge symmetry. For a detailed history of Kaluza-Klein theories, see O'Raifeartaigh and Straumann (1998b), and for a recent proposal of modern Kaluza-Klein theory see e.g. (Horoto and Scholtz 2024).

[60] All quoted text in this section are extracted from the reproduction and translation of the original works of Kaluza and Klein found in O'Raifeartaigh (1997) p.53 to p.69.

### 7.5 Concepts: Dirac equation and the gauge covariant derivative

In a sense, relativistic quantum electromagnetism[61] can be seen as a straightforward generalization of the non-relativistic quantum mechanics, in which the Schrödinger equation (17) is replaced by the so-called Dirac equation:

$$\boxed{(\mathbb{i}\gamma^\mu\partial_\mu - m)\psi = 0} \tag{39}$$

Here $\psi$ is a more complicated object than $\psi$ as it is a spinor, which means that it has multiple components (4 for the Dirac equation) and transforms in a specific way under Lorentz transformations. The $\gamma^\mu$ are $4 \times 4$ matrices called the gamma matrices, which are used to write the Dirac equation in a covariant form and are associated to basis vectors of the space-time.

One can notice that the Dirac equation is invariant if the field is transformed as

$$\psi \to e^{\mathbb{i}\epsilon}\psi, \tag{40}$$

where $\epsilon \in \mathbb{R}$ is a constant real number. This transformation, known as a phase transformation (and its corresponding symmetry is the phase invariance), is already present for the wavefunction in quantum mechanics. It translates the fact that the probability to observe a particle at the point $x$ is given by $P(x) = \psi^*\psi$, which is independent on the phase of $\psi$. As such, phase invariance translates that the phase of the wavefunctions are not observable as they do not change the probabilities and hence the predictions of quantum mechanics. The phase invariance Eq. 40 is often referred to as a **global** transformation as it consists of changing the phase of $\psi$ by the same amount at every point of space-time. While the link with gauge transformations seems a very far reach, we will make it clear below.

One can further notice that the Dirac Lagrangian is not invariant anymore if $\epsilon$ becomes a function of $x$ and $t$ i.e. if we change the phase of $\psi$ differently at different space-time points:

$$\psi \to e^{-\mathbb{i}\Lambda(x^\mu)}\psi, \tag{41}$$

where the minus sign is chosen purely by convention. While Eq. 40 was reffered to as a global transformation, Eq. 41 is instead reffered to as a **local** transformation. We already encountered a similar transformation in Eq. 22. Under such a transformation, the derivative operators $\partial_\mu$ will indeed act on $\Lambda(x^\mu)$ and lead to spurious and unwanted terms, changing the form of the Dirac equation.

Now, the so-called **gauge argument** goes this way: the Dirac equation is invariant under a global phase transformation, it should also be modified to become invariant under a local phase transformation. As we saw in the previous section, such invariance can be asked for by asking for the theory to be local. Indeed, as a consequence of special relativity, no information should be able to travel faster than the speed of light (we will re-examine in Part II whether this motivation for the gauge argument is sensible). Asking for this requirement, one can argue that the free Dirac equation must be modified in a way that make it invariant under transformations of the form Eq. 41 as:

$$\boxed{(\mathbb{i}\gamma^\mu D_\mu - m)\psi = 0} \tag{42}$$

which require the introduction of a new field $A_\mu$ in the covariant derivative as

$$D_\mu = \partial_\mu + \mathbb{i}qA_\mu \tag{43}$$

This new form for the Dirac equation, generalizes Eq. 20. The new field $A_\mu$ must transform along with $\psi$ under a local phase transformation as

$$\begin{cases} \psi \to \psi' = e^{-\mathbb{i}\Lambda}\psi \\ A_\mu \to A'_\mu = A_\mu + \partial_\mu\Lambda \end{cases} \tag{44}$$

in order to ensure the invariance of the Lagrangian. One would then recognize Eq. 14 and identify $A = (V, \vec{A})$ as the electromagnetic potentials.

[61] Here for most particles composing matter, that is fermions as electrons and quarks. Note that everything discussed here will remain valid for bosons and the Klein-Gordon equation.

The Minimal coupling rule (Eq. 21), is given by a single covariant equation:

$$\partial_\mu \to \partial_\mu + \mathbb{i}qA_\mu, \tag{45}$$

which simply states that regular derivative must be replaced by covariant ones. Conservation of charge is encoded naturally through the so-called Noether theorem (already present in classical electrodynamics but a bit harder to derive (Mieling 2017)).

This provides remarkable insights: The gauge field $\vec{A}$ and the interaction it mediates (here again apparently more fundamental than $\vec{E}$ and $\vec{B}$ which are not needed in the formulation), seems to appear as a necessity from symmetry considerations, looking like the electromagnetic interactions between particle is "required" from the gauge argument. Gauge transformation of classical electromagnetism is a natural consequence of the global phase invariance of the Lagrangian and electric charge is the Noether charge associated to this invariance.

# 8 Potentials in Yang-Mills theories and modern particle physics

## 8.1 History: the gauge principle as a guiding principle for the construction of all interactions

Weyl's procedure for generating an interaction from a gauge symmetry was only gradually assimilated by physicists. Although his 1929 reformulation introduced the essential idea that interactions arise from the requirement of local symmetry invariance, this insight remained largely disconnected from mainstream particle physics for more than two decades. Its later generalization would ultimately provide the framework used to describe all known fundamental interactions.

An important precursor emerged in nuclear physics. Following the discovery of the neutron, Heisenberg (1932) proposed that protons and neutrons should be regarded as two states of a single particle, related by an internal symmetry. In analogy with ordinary spin, this property was later formalized by Wigner (1937) as isotopic spin (isospin), described mathematically by transformations of the group SU(2). At this stage, however, isospin symmetry was understood as a global symmetry relating particle states rather than as a dynamical principle generating interactions.

The decisive conceptual step was taken by Yang and Mills (1954), who proposed extending isospin from a global to a local symmetry. Generalizing Weyl's gauge principle to a non-commutative group, Yang and Mills introduced gauge transformations acting on the proton–neutron doublet through the group SU(2). Requiring invariance under local isospin rotations forced the introduction of new vector fields, thereby producing an interaction determined entirely by symmetry principles. This construction marked the birth of non-Abelian gauge theory. Historically however, Yang and Mills did not explicitly trace their work back to Weyl's work (which is not cited); and instead considered Heisenberg 1932's previous work on isospin as well as Pauli 1941 work on relativistic fields. As such, the motivations leading to the proposal of Yang and Mills has now become deeply disconnected from to the geometrical ambitions of the first explorations of Weyl, Kaluza and Klein[62] [63].

A similar and independent derivation to the one of Yang and Mills was proposed in his PHD thesis by Ronald Shaw (1955)[64], in a proposition to generalize the U(1) symmetry of spinors – understood as isomorphic to a SO(2) invariance [65] – to Weinberg's isospin group SU(2).

Shortly thereafter, Utiyama (1955) provided a systematic formulation by showing how continuous Lie groups $G$ can, in principle, be promoted from a global symmetry to a local one, thereby generating

[62] As discussed in Chap.8 of O'Raifeartaigh 1997, Yang admitted that there motivations were "completely divorced from general relativity". Even more curiously, Yang and Weyl spent several years together at Princeton University where they interacted, and yet they never discussed physics or mathematics, despite their clear and profound shared interests.

[63] Just as in electromagnetism, attempts to construct higher-dimensional theories of the nuclear forces were made by Oskar Klein as early as 1939, and later by Wolfgang Pauli in 1953. These theories share similarities with modern gauge theories in several aspects and in this sense, higher-dimensional approaches once again anticipated the later, four dimensional developments. Although Pauli's formulation was highly promising, he ultimately abandoned it because he could not identify a mechanism to generate masses for the gauge bosons. On this topic, see Straumann (2000) and Chapters 6 and 7 of O'Raifeartaigh (1997).

[64] See the reproduction in Chapter 9 of O'Raifeartaigh 1997.

[65] We shall come back to the isomorphism between U(1) and SO(2) in Sec. 13.3.

interaction fields. Weyl's gauge principle thus sees itself fully generalized to any group. Utiyama "find(s) an analogy between the transformation character of the electromagnetic field $A_\mu$, the Yang-Mills field $B_\mu$ and the Christoffel's affinity $\Gamma^\lambda_{\mu\nu}$ in the theory of the general relativity" (p.1597 of Utiyama (1955)). However, here again, Weyl's work are not cited and while the analogy between general relativity, electromagnetism and Yang-Mills theories is clearly identified, Utiyama do not provide a geometrical reading of gauge theories. It is even quite the opposite. Since "the covariant derivative of any tensor or spinor can be derived from the postulate of invariance under the "generalized Lorentz transformations" [...] deriving such covariant derivatives it is unnecessary to use explicitly the notion of parallel displacement" (Utiyama (1955) p.1598). As such, Utiyama's approach might be better understood as making geometry superflous in general relativity instead of geometrizing gauge theories.

The Yang–Mills framework was initially met with skepticism, largely because the resulting gauge bosons were necessarily massless, whereas the strong and weak nuclear interactions were known to be short-ranged. Despite this difficulty, the gauge principle proved too powerful to abandon. During the early 1960s, it was adapted—independently and in different physical contexts—by Glashow (1961), Weinberg (1967), and Salam (1969), who proposed that electromagnetic and weak interactions could be unified within a single SU(2)×U(1) gauge theory. In this electroweak framework, the photon and the weak vector bosons emerge from a common gauge structure, extending Weyl's phase symmetry into a unified description of interactions.

The modern theory of the strong interaction followed a different path. Although early Yang–Mills models were based on isospin SU(2), developments in hadron spectroscopy led Gell-Mann (1964) and independently Zweig (1964) to propose the quark model, introducing an additional internal degree of freedom later called color. The strong interaction was subsequently formulated as a gauge theory based on the group SU(3), giving rise to quantum chromodynamics (QCD), in which gluons act as the gauge bosons mediating interactions between quarks.

At first sight, the weak interaction appeared incompatible with gauge symmetry because its mediating bosons, $W^\pm$ and $Z_0$, possess nonzero masses. A major theoretical breakthrough occurred in 1964 with the realization that gauge invariance can be preserved through spontaneous symmetry breaking. Independently, Englert and Brout (1964), Higgs (1964), and Guralnik, Hagen, and Kibble (1964) demonstrated that introducing a scalar field with a non-vanishing vacuum expectation value allows gauge bosons to acquire mass. This mechanism, later incorporated into the electroweak theory, completed the modern gauge-theoretic description of particle interactions. The associated scalar particle—the Higgs boson—was experimentally discovered at the Large Hadron Collider in 2012 (Atlas Collaboration 2012; CMS Collaboration 2012), providing striking confirmation of a framework whose conceptual roots can ultimately be traced back to Weyl's gauge principle.

As such, while the origins of gauge theories were deeply rooted in the desire to geometrize electromagnetism in the aftermath of general relativity, the developpements ranging from 1930 to 1970 were instead generalizing (consciously or not) Weyl's gauge principle in a purely abstract fashion. As stated in Sec. 7.3, the required tools for a modern geometric reading of gauge symmetry (fiber bundles) were developed in parallel by authors such as Cartan (1926) and Ehresmann (1951) from purely mathematical considerations about the nature of geometrical spaces (even though general relativity had a major influence on such developments). Much later works as Trautman 1970 and Wu and Yang 1975a[66] will make the connections between these two approaches and allow for the modern geometrical reading of gauge theories which will be the topic of Part III.

## 8.2 Concepts: Yang-Mills and non-Abelian gauge theory

We are now ready for a general presentation of gauge theories in the context of particle physics. This will allow us to introduce or re-introduce most of the terminology required for discussions in the rest of this essay.

Let $\psi$ be a matter field, that is a spinor field describing one way or another the possible presence of fermions at each point of space-time. In general, $\psi$ describes what can be identified classically as

[66] See in particular Table I p.3851 which provides a "translation of terminology" between the "gauge field terminology" and the "bundle terminology".

"matter": electrons and quarks. $\psi$ is in general a multi-component objects made of several spinors[67] $\psi^i$. We will ignore those matter indices in the following, in order to keep the notations reasonable.

A gauge theory is associated to a Lie group $G$, which is called the **gauge group** or **structure group** and a specific representation of this group $\rho(G)$ must be able to act on $\psi$. We write $X_\alpha$ the generators of $G$ in the $\rho$ representation. There are as many gauge fields $A^\alpha_\mu$ as there are generators of $G$. The gauge fields $A^\alpha_\mu$ are to be understood classically as the mediator of the forces: photons, gluons and bosons $Z$ and $W$. The intensity of the gauge interaction between $\psi$ and the $A^\alpha_\mu$ is set by a constant known as the **gauge coupling constant** $\bar{g}$ which is a fundamental constant of nature.

We further introduce the so-called **covariant derivative** which describes the interaction between the matter field and the gauge field as

$$\boxed{D_\mu = \partial_\mu + \bar{g} A^\alpha_\mu X_\alpha,} \tag{46}$$

which generalizes Eq. 43. The **minimal coupling rule** (Eq. 21 and 45) is the replacement $\partial_\mu \to D_\mu$ in the free Dirac equation (Eq. 39). The **field strength tensors** which describe the dynamics of the gauge fields and their self-interactions as

$$\boxed{F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + \bar{g} f^\alpha_{\ \beta\gamma} A^\beta_\mu A^\gamma_\nu,} \tag{47}$$

where $f^\alpha_{\ \beta\gamma}$ are the structure constants of the Lie Algebra $\mathfrak{g}$ appearing in the commutation relation between the generators $[X_\beta, X_\gamma] = f^\alpha_{\ \beta\gamma} X_\alpha$. One would easily recognise here the generalization of Eq. 32.

We mentioned already several times that modern theories were written in terms of minimisation of an action, which is the integral of a Lagrangian function. Lagrangian play such an important part that they are often used as a way to define a theory. Since defining gauge theories is one of the challenge of this essay, let us explicit their Lagrangian here. A **gauge theory** is then a theory which is described by the Lagrangian density[68]

$$\mathcal{L}_{\psi A} = \overline{\psi} \left( \mathbb{i} \gamma^\mu D_\mu - m \right) \psi - \frac{1}{4} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha, \tag{48}$$

where $m$ is the mass of the matter field and $\gamma^\mu$ are four symbols usually represented as $4 \times 4$ matrices known as the Dirac matrices and $\overline{\psi} = \psi^\dagger \gamma^0$ is known as the Dirac adjoint of $\psi$. We see from these equations that Yang-Mills theory really generalize Maxwell theory, to more complex situations (more precisely, to any gauge groups $G$).

The equation of motion of gauge theories are obtained by extremalization of the action (space-time integral of $\mathcal{L}_{\psi A}$) and are:

$$\begin{cases} \left( \mathbb{i} \gamma^\mu D_\mu - m \right) \psi = 0, \\ D_\mu F^{\mu\nu}_\alpha = \bar{g} J^\nu_\alpha, \end{cases} \tag{49}$$

where $J^\mu_\alpha = \overline{\psi} \gamma^\mu X_\alpha \psi$ is the **charge current**. The first equation, known as **Dirac equation** describes the evolution of the matter field $\psi$ subject to the force exerted on it by the gauge fields. The second equation, known as **Yang-Mills equation** describes the evolution of the gauge fields themselves under the influence of matter fields which generate them. In this equation, we noted the covariant derivative of the strength tensor $D_\mu F^{\mu\nu}_\alpha = \partial_\mu F^{\mu\nu}_\alpha + \bar{g} f^\gamma_{\ \alpha\beta} A^\beta_\mu F^{\mu\nu}_\gamma$. These two equations are structurally identical to the electromagnetic Dirac equation (Eq. 42) and charged Maxwell equations (Eq. 35).

Similarly, after a bit of work it is possible to show from the properties of $F$ that:

$$\boxed{D_\mu F^\alpha_{\nu\rho} + D_\nu F^\alpha_{\rho\mu} + D_\rho F^\alpha_{\mu\nu} = 0} \tag{50}$$

which generalizes Maxwell's charge-free equations (Eq. 34). Similarly, one can derive the Yang-Mills continuity equation $D_\mu J^\mu_\alpha = 0$, translating charge conservation.

[67] Each spinor $\psi^i$ composing $\psi$ being itself a multi-component object!

[68] Here and in the reminder of this work, Lagrangian densities are related to the action to extremize as $\mathcal{S} = \int \mathcal{L}\mu \ \mathrm{d}^4x$ where $\mu = \sqrt{|g|} = e_0 \wedge e_1 \wedge e_2 \wedge e_3$ is the volume element and $g$ is the metric tensor.

When applied to specific gauge groups in the context of high energy physics, these theories are often named **Yang-Mills** theories. Under the choice of the gauge group $G =$U(1), this theory describes electromagnetism. The gauge coupling constant of electromagnetism is named the elementary charge and is noted $e$. Matter fields $\psi$ must live in vector spaces on which the representations of U(1) can act. The irreducible representations $\rho_q$ of U(1) are labelled by a constant number $q \in \mathbb{Z}$ called the electric charge. They all act on one dimensional complex vector space isomorphic to $\mathbb{C}$. If $\psi \in \mathbb{C}$ is a member of this vector space, it transforms as $e^{\mathbb{i}q}\psi$ under a U(1) transformation and it is said that $\psi$ has an electric charge of $q$. There is only one generator of U(1), which is $X = \mathbb{i}q$ and thus one gauge field $A_\mu$ known as the **photon**. If one expresses all electric charges in units of $e$, one recovers all the equations of covariant electromagnetism given in Sec. 6.

For the choice of the gauge group $G =$SU(2), this theory describes weak theory at high energies. To the three generators of the group are associated three gauge fields, known as the $W$ **bosons**. The matter fields are quark doublets or electron/neutrinos (lepton) pairs. Both forces can be unified in the so-called electroweak theory of gauge group SU(2)×U(1) under the addition of a scalar field known as the Higgs field, giving mass to the matter fields and some of the gauge fields[69]. With the gauge group SU(3), this theory describes strong interaction within quantum chromodynamics. The matter fields $\psi$ are triplet of colored quarks (red, green and blue) and the gauge fields are the eight **gluons** associated to the eight generators of SU(3). Quantum chromodynamics and the electroweak theory form a gauge theory of the group U(1)×SU(2)× SU(3) known as the **standard model of particle physics**. In its quantized form, it describes all the known fundamental interactions of nature besides gravity.

Let $h(x) = e^{X_\alpha \theta^\alpha(x)}$ be the given of an arbitrary (and different) representation of an element of the gauge group $h \in \rho(G)$ at every space-time point $x$. The two equations of motion of the theory presented in the previous section are invariant under the joint transformation of $\psi$ and $A^\alpha_\mu$:

$$\begin{cases} \psi \to \psi' = h\psi, \\ A^\alpha_\mu X_\alpha \to (A^\alpha_\mu)' X_\alpha = h^{-1} A^\alpha_\mu X_\alpha h - h^{-1}\partial_\mu h. \end{cases} \tag{51}$$

This pair of transformations is called a **gauge transformation**. The first equation is sometimes named a **gauge transformation of the second kind** while the second is a **gauge transformation of the first kind**. The field strength also transform under a gauge transformation, as $F^{\mu\nu}_\alpha \to hF^{\mu\nu}_\alpha h^{-1}$. The symmetry of the gauge theory under the transformation Eq. 51 is the general expression for **gauge symmetry** or **gauge invariance**. While all the empirical predictions of the theory remain unchanged under a gauge transformation, the values of $\psi$ and $A^\alpha_\mu$ change to $\psi'$ and $(A^\alpha_\mu)'$. As such, these quantities are said to be **gauge dependent**. One could consider any arbitrary function $h(x)$ and there are infinitely many $\psi$ and $A^\alpha_\mu$ related by the transformation Eq. 51. All these $\psi$ and $A^\alpha_\mu$ pairs are said to be located on the same **gauge orbit**. In practice, one can always use a specific representative of gauge field within the gauge orbit in a way that simplify specific calculations for given problems. Doing so is called **gauge fixing** or simply making a **choice of gauge**.

## Part II

# A thorny challenge: How to interpret gauge theories?

Interpreting gauge theories is a first importance and highly non trivial task. Highly important as gauge theories are giving our most accurate description so far of the fundamental interactions of nature and non trivial as they rise multiple difficult questions and involve advanced mathematical tools.

As emphasized by Guttmann and Lyre (2000), the question of interpretation becomes particularly pressing in the context of gauge theories, where multiple mathematically distinct formulations may

[69] The addition of the Higgs field opens the more complex topic of spontaneous symmetry breaking which would deserve a whole philosophical investigation by itself. We will not further discuss it here and refer to Berghofer, François, Friederich, et al. (2023) and Friederich (2011).

represent the same physical situation. This issue, also discussed in Field (2016), highlights the tension between the indispensable role of mathematical structures in physical theories and the challenge of identifying which elements correspond to genuine physical content rather than representational redundancy.

For now, we consider only classical gauge theories, in which $\psi$ and $A$ are understood as relativistic fields and not as quantum fields (in a sense matter is treated as a quantum field, but not the mediator field). We will not discuss the special status of general relativity, which will instead be a central topic of the next part. We will simply assume that classical gauge theories are empirically validated, and see if there is any lesson we can learn from them.

Gauge theories raise a number of questions regarding their interpretation. To list only a few of them: what entities of gauge theories are to be considered as physical? how to think about gauge symmetry in regard of the other symmetries of a theory? Are gauge transformations and the gauge argument devoid of physical content? Can other theories, as general relativity, be considered as gauge theories and in what sense? (the discussion of this last point is left for Part III). In Sec. 9 we will discuss where gauge symmetries stand compared to the other more familiar symmetries of physical theories. In Sec. 10, we will discuss the significance of the gauge argument. In Sec. 11, we will remind ourselves of the Aharonov-Bohm effect and discuss its consequences and generalisations. Finally, in Sec. 12.1 we will discuss the main possible roads to interpret classical gauge theories. For general reviews on the philosophical issues raised by gauge theories, see also Dougherty (2025), Guay (2013), Healey (2007), Lyre (2004), and Redhead (2002).

# 9 How do gauge symmetries compare to other symmetries?

In physics, symmetries undoubtedly play a key role. However, understanding exactly this role and what is meant by symmetry is far from an easy task (see e.g. Brading, Castellani, and Teh 2023; Teh 2024). Intuitively, a symmetry corresponds to some operation (physical on a system and/or mathematical on an equation) that leaves a physical situation unchanged. If I take a ball and let it fall on the floor and I repeat this exact same operation a few meters away on my right, the ball will fall identically (all else being equal). Such a situation depicts invariance under spatial translations, and similarly the equations of motions for the ball will remain the same, whether considered here or a few meters on the right. To be a bit more accurate, we should first introduce two very important concepts which are crucial in the discussion of symmetries. If a physical equation/law keeps the same form after a transformation, it is **covariant**. If a quantity remains numerically identical, it is **invariant**. As such, Eq. 35 is covariant and the speed of light $c$ is invariant under boost/Lorentz transformations from one inertial frame to another. Neither Newtonian mechanics nor special relativity are covariant under arbitrary frame transformations, but general relativity is. Then, one must distinguish active and passive transformations:

- **An active transformation/symmetry**: An active transformation is the physical transformation of a system into another one. For example, rotating a ball, switching two particles etc If all the empirical predictions of a theory are invariant under such a transformation, it is a symmetry.

- **A passive transformation/symmetry**: A passive transformation is a mathematical change in the description of a unique physical phenomenon. It could be the change of units in which to express temperature from Celsius to Fahrenheit. It is therefore better understood as a redescription rather than as a physical symmetry. If they are not symmetries of the theory, the theory is not covariant.

In a given formalism, an active transformation often has an active counterpart. For example, consider a vector in classical mechanics, written in a basis as $\vec{v} = v^i \vec{e}_i$. Let $R$ be a rotation matrix. In an *active* rotation, the basis is kept fixed and the vector is rotated: $\vec{v} \mapsto \vec{v}' = R\vec{v}$, and so $v'^i = R^i{}_j v^j$. In the corresponding *passive* rotation, the vector is kept fixed while the basis is rotated: $\vec{e}_i \mapsto \vec{e}_i^{\,'} = R\vec{e}_i$, and so $v'^i = (R^{-1})^i{}_j v^j$, so that $\vec{v} = v^i \vec{e}_i = v'^i \vec{e}_i^{\,'}$. Depending on conventions, one may write the basis transformation with $R^{-1}$ instead of $R$; the essential point is that, in a passive transformation, the basis and the components transform inversely so that the vector itself is unchanged.

Secondly, one must distinguish global and local symmetries:

- **A global symmetry**: A global symmetry is the invariance of all the empirical predictions of a theory if an operation is done identically across all space-time. For example the addition of a constant number to $V$ is a global symmetry of Newtonian theory presented in Sec. 2 and so is the multiplication of the matter field by a constant phase (Eq. 40), which leave probabilities unchanged in a quantum context. Through the so-called Noether theorem, to each global symmetry is associated a conserved quantity. This is the case of energy for invariance in time translation, linear momentum for invariance under space translations, angular momentum for invariance under rotation and charges under phase/global gauge symmetries.

- **A local symmetry**: A local symmetry is the invariance of all the empirical predictions of a theory if an operation is done differently at different points of space-time. This is the case of the local phase invariance (Eq. 41).

Symmetries play a central role in model building and in the development of new theories. The canonical example is that of Galileo's ship: any experiment performed aboard a ship moving at constant velocity yields the same results as it would at rest on Earth (this is invariance under boost transformation). If one were confined below deck, without any window, it would be impossible to determine whether the ship is at rest or in uniform motion. This insight was later generalized by Einstein — who preferred trains to ships — ultimately leading to the formulation of special relativity.

Now, what does gauge symmetry have to do with this?

Gauge invariance as formulated by Weyl 1929b undoubtedly played and still plays a critical role in model building. Its generalisation to other forces than electromagnetism led to the derivation of Yang and Mills 1954 and imposing some form of gauge invariance to a general theory was clearly depicted as a powerful tool to model new elementary forces in Utiyama 1955. Today, it is common practice to propose new particles or force fields beyond the Standard Model by introducing additional gauge degrees of freedom. This paradigm remains central to the quest for Grand Unified Theories (GUTs), which aim to merge disparate interactions into a single, overarching gauge symmetry group. Furthermore, frameworks such as supersymmetry and string theory represent the ultimate extension of this logic, treating gauge symmetry not merely as a mathematical convenience, but as a fundamental requirement for the consistency of quantum gravity and the unification of all force/matter fields in a single formalism. The contemporary status of gauge symmetry has undergone a conceptual shift: it is now viewed less as a mathematical remedy to accommodate empirical data—as was the case for Weyl and Yang-Mills—and more as a proactive structural principle. In modern model-building, unless a symmetry is explicitly forbidden by mathematical inconsistency or quantum anomalies, it is treated as a viable, and perhaps even necessary, extension of the theory. This is most clearly seen in Supersymmetry, where a symmetry relating bosons and fermions is proposed on the ground that it is possible to do so, not on the observation of some super-partners.

Now, it seems that it is impossible to perform a Galileo-ship kind of experiment regarding gauges. A gauge transformation does not relate two empirically distinct situations in the way a spatial translation or a boost does. Rather, it connects different mathematical representations of what is taken to be the same physical situation. As discussed in Greaves and Wallace 2014, Teh 2016, Gomes 2019 and Ramírez and Teh 2025 the situation regarding gauge transformation might however be subtler than it might appear. Taken together, these papers argue that the empirical significance of gauge symmetry can only be properly understood once one distinguishes genuine descriptive redundancy from physically meaningful subsystem transformations and takes boundary structure seriously. A boundary is the location where the subsystem under studying ends. It is not a vague concept as it might seems, and plays a crucial role in interpreting gauge (See also the discussion in Rovelli 2020). In particular, Teh 2016 argues that asymptotically non-trivial transformations that preserve the appropriate boundary conditions can yield a Galileo's-ship-style, relational form of direct empirical significance, by changing the relation between a subsystem and its environment without amounting to a mere redescription. However, as argued by Ramírez and Teh 2025, purported non-relational cases of empirical significance are better understood not as cases of gauge symmetry proper, but as involving rigid boundary symmetries associated with boundary charges.

# 10 Status of the gauge argument

The gauge argument, which was presented in Part I, seems to indicate that requiring phase invariance of the matter field $\psi$ induces the emergence of interactions. As we saw, the gauge argument itself was introduced by Weyl 1929b, and the name was coined by Earman 2002 As discussed in the previous section, the gauge argument was and is now extensively used in order to introduce new fields in physical models. Its generalisations are so far reaching that Earman 2002 describes it as "the most fundamental cornerstone of modern theoretical physics", so it is critical to have a closer look at it. Let's define its structure as follows:

- The theory posess some kind of global symmetry, generally at the Lagrangian level.
- Information cannot travel faster than light because of special relativity. As such, this symmetry should be encoded at each point of space-time. The global symmetry is promoted to a local one.
- The original theory is not covariant under the proposed local symmetry. This can be solved by introducing a new field, the gauge field, which transform in a proper way (gauge transformations of the first kind) in order to compensate new terms introduced by the local symmetry under a transformation. To do so, the new field must be introduced in a way known as minimal coupling.
- The gauge field is the mediator of interactions. So interactions emerge from local invariance.

This presentation is schematic and a bit caricatural, but reflects correctly how numerous textbooks introduce the gauge argument or gauge principle, and thus how many practicing physicist learned to think about the significance of gauge. See for example Ryder 1996. Note that, such argument do not require the quantum context, and a similar "gauge argument" can be derived in the context of classical electromagnetism (Kobe 1980).

Now, we can list at least three major problems with the gauge argument:

- First, none of the points in the argument above are strict requirements or logical steps. They are, at best, suggestions. Indeed, for example it is possible to propose different replacements of the derivative than minimal coupling in the equations, or other transformations, which allow to preserve the invariance of the theory. On this point, see Barlow et al. 1993 and Heras 1994. The minimal coupling rule can be more easily motivated by the classical form of the Hamiltonian for the Lorentz force, injected within Schrödinger equation through the correspondence principle[70].
- Secondly, the gauge argument is motivated by the fact that the theory must be invariant under a transformation done locally and differently at each point of space-time for some locality reason. Usually it goes as follows: if one were to do a global transformation of the field all at once in the Universe (Eq. 40), the information would have to propagate faster than light. However, such an argument, while very satisfying at first, turns out to be confusing when thinking about it. What transformation is actually done on the field and by whom? This suggests some unclear active interpretation of gauge theory.
- Thirdly and crucially, the gauge argument do not say anything about the field or it's dynamics. It does not imply anyhow the existence of the Maxwell equations and their generalisation. It does not say anything about the charge either. Now, as Wallace 2009 points out: how is this argument supposed to work, for example for neutral particles? Here the coupling with the gauge field must be zero. Nothing in the gauge argument can encode that. As such stating that the interaction emerge from the requirement of local phase invariance appears as an overstatement.

As such, we might be tempted to discard completely this argument as void of physical significance. We will however see that the gauge invariance can be associated with what we consider as a much more plausible geometrical interpretation. We will come back to it in Sec. 14.4. Note also that recent works,

[70]Doing so reverses the logic of the gauge argument and local phase invariance of $\psi$ becomes a consequence of the electromagnetic gauge invariance on $V$ and $\vec{A}$.

as Gomes, Roberts, and Butterfield 2022 provided motivation as well as a more rigorous formulation of the gauge argument in the context of Noether's second theorem, allowing to consider it as a constraint on the allowed way in which charges can couple to fields. As such, this gives back a bit of the argument's prestige and allow us to understand why such an argument can play such a powerful role in terms of model building.

## 11 Underdetermination and the Aharonov-Bohm effect

The most critical philosophical problem underlying gauge theory is the **underdetermination** carried by the gauge potentials. We saw that, the gauge fields $A^\alpha_\mu$ play a key role in the formulation of gauge theories and are tempting candidates to be considered as physical quantities. However, an infinite number of them, all related through gauge transformations can represent the same physical reality. No observation whatsoever could isolate and distinguish between two $A^\alpha_\mu$ lying on the same gauge orbit. As discussed in Part I (Sec. 5.2.2), the Aharonov–Bohm effect marks a decisive shift in the interpretation of gauge theories. In classical electrodynamics, gauge potentials can be treated as auxiliary mathematical constructs, introduced for calculational convenience, while only the field strengths carry direct physical significance. The AB effect challenges this view. It shows that electromagnetic potentials can produce observable consequences even in regions where the corresponding force fields vanish.

In the notation introduced earlier, the effect demonstrates that a matter field $\psi$ can be influenced by a gauge field $A^\alpha_\mu$ in a spacetime region where the field strength tensor $F^\alpha_{\mu\nu}$ is identically zero. The influence is encoded not in the local value of $F^\alpha_{\mu\nu}$, but in the global properties of the gauge connection, specifically in the nontrivial holonomy associated with the enclosed flux strength flux. We use notations for general gauge theories here, as the philosophical implications of the AB effect extend beyond electromagnetism. Indeed, analogous effects can, in principle, arise in any gauge theory, although its physical consequences may not always be directly observable. This includes non-Abelian gauge interactions such as those describing the strong force, as well as gravity when formulated in gauge-theoretic terms. The phase difference acquired by the wavefunction on the closed path $\gamma$ is given by the so-called **holonomy** or **phase factor**

$$\mathfrak{h}(\gamma) = \mathcal{P} \exp\left( - \oint_\gamma A^\alpha_\mu X_\alpha \cdot \, \mathrm{d}x^\mu \right), \tag{52}$$

where $\mathcal{P}$ denotes a path-ordered exponential, the angle of rotation acquired by a vector transported back to itself on the loop. In quantum field theory, the gauge-invariant observables constructed from such holonomies are known as **Wilson loops**[71]. They play a central role, for instance, in the non-perturbative characterization of confinement in non-Abelian gauge theories. Closely related effects have also been identified in the gravitational context, where phase shifts arise from spacetime curvature in interferometric setups (Overstreet et al. 2022).

Skepticism toward the reach of the AB effect is understandable. A common objection is that the idealized conditions required for the AB setup—most notably an infinitely long solenoid or perfectly confined magnetic flux—are never strictly realized in practice. One might therefore question whether the effect genuinely occurs in a region free of magnetic field.

However, this objection is empirically unfounded. The experimental realization by Akira Tonomura et al. 1982 and collaborators employed a carefully engineered toroidal ferromagnet that confined the magnetic field within its core. The interfering electrons propagated exclusively through regions where the field strength was strictly null, yet the observed phase shift agreed with the theoretical prediction (Eq. 24). The experiment therefore confirms that the relevant physical effect arises from the gauge potential's global structure rather than from local field strengths.

[71]In non abelian gauge theories $\mathfrak{h}(\gamma)$ is not gauge invariant anymore. However, its trace is. This trace is the definition of a Wilson loop.

# 12 Interpretations of gauge theories and their philosophical consequences

## 12.1 Philosophical concepts threatened by gauge theories

Perhaps one of the most pressing questions raised by gauge theories is to identify, in light of the gauge transformations, what bears the ontological content of the theory and what is to be considered as mathematical surplus or redundancy. Quite interestingly, the existence of phenomena as the AB effect deprives us of any simple "straightforward" interpretative framework, such that we are constrained to draw a picture of the world from gauge theories which will have to go against our classical intuition one way or another. Indeed, choosing an interpretative framework for gauge theories always comes at a philosophical price. It is this price that we will try to assess in the coming sections.

A first concept which is challenged by the AB effect is the one of locality. However, as discussed by Lyre (2004) or Healey (2007), different notions associated to the concept of locality should be considered to discuss this effect (see also Eynck, Lyre, and Rummell (2001)). In this work, we consider the following:

- An interpretation is said to contain **point-wise interactions** if the different entities involved interact with each other directly in arbitrary small regions idealized as space-time points in which they are both present (i.e. non-zero)[72]. As an example, let's consider classical electromagnetism of a point particle and interpret $\vec{E}$ and $\vec{B}$ as physical ontological bearing quantities. We dub this interpretation the standard interpretation of classical electromagnetism. In that framework, the Lorentz-force (Eq. 5) is typically a point-wise interaction. $\vec{E}$ and $\vec{B}$ induce an acceleration to a particle located at a point of space $x$ if and only if they take non-zero values at this point $x$.

- The interpretation has **local actions** if every event is caused by some continuous propagating physical process. Put differently: if $A$ and $B$ are spatially distant physical systems, an alteration of the state of $A$ cannot have an immediate consequence on the state of $B$. This notion is very similar to but can be distinguished to the more formal **relativistic locality** stating that "all the causes of any event lie inside or on its back light-cone"(Healey 2007). Relativistic locality states that all events and causes could at least in principle be related by a local action which velocity of propagation is bounded by the speed-of-light. For what concern us here, we will identify the two concepts under the term of local actions. The Lorentz force in the standard interpretation of classical electromagnetism is a local action in both senses as perturbations of $\vec{E}$ and $\vec{B}$ have to propagate from one point to another in space at the speed-of-light according to Maxwell equations (Eq. 6).

- An interpretation displays **separability** if "Given a physical system $S$ and its exhaustive, disjoint decompositon into subsystems, it is possible to retrieve the properties of $S$ from the properties of the subsystems."(Lyre 2004). The term **holism** is frequently used to describe non separable interpretations. Clearly again, the Lorentz force acting on a point charge is separable as we can understand all the properties of the moving particle from the properties of the subsystems at play which are the electric field, the magnetic field and the particle itself.

As discussed above, standard interpretation of classical electromagnetism satisfies all of these criteria. On the other hand, it is clear that standard quantum mechanics with the Copenhagen interpretation does not satisfy any of these criteria due to entanglement. As we will discuss, the existence of the AB effect will force us to make some sacrifices on these conceptions of locality in the context of classical electromagnetism applied on a wavefunction (and on any gauge theory for that matter) even without having to invoke any form of entanglement.

Other conceptual notions are important and questioned by gauge theory

- **Measurability**: If an interpretation is based on one or several ontological entities, one can wonder whether these ontological bearers can be measured directly or not. As one can measure

[72] "Interacting entities can be defined within arbitrarily small spacetime regions (usually idealized as spacetime points), couple to each other in that region and are non-zero in overlapping regions." (Lyre 2004)

both $\vec{E}$ and $\vec{B}$ using an electrostatic fieldmeter and a magnetometer respectively, the standard interpretation of classical electromagnetism is measurable.

- **Determinism**: An interpretation will be said to be determinist, if by knowing the value of the relevant quantities at a given time $t_0$, It is possible to infer uniquely their values in the future or in the past using the equations of motion of the theory. Standard interpretation of Classical electromagnetism is deterministic as it is possible to infer uniquely the values of $\vec{E}(t)$ and $\vec{B}(t)$ at any point in time $t$ using Maxwell equations by knowing their exact values at a time $t_0$.

Another notion that is omnipresent in the discussions about gauge theories is the notion of **surplus structure** or **excess structure** (Earman 2002; Redhead 2002) The question here is whether or not the formulation of a theory contains mathematical redundancy. Due to the necessary use of gauge variant quantities, gauge theories as formulated in Sec. 8.2 are the paradigmatic example of a theory containing excess structure. More generally, any theory containing symmetries i.e. transformations which keep its predictions unchanged, will unavoidably contain some form of redundancy and excess structure. While it might seem like a pretty straightforward task, the notion of symmetry is extremely difficult to define in full generality as well as to interpret. Digging too deeply around these lines would lead us too far and we refer to Belot (2011) and Brading, Castellani, and Teh (2023) for some discussions. As such, it is extremely relevant to see how gauge symmetry compares to the other symmetries in physics and whether they have empirical consequences. On this topic, see e.g. Brading and Brown (2004) and Greaves and Wallace (2014). Additionaly, the question of excess structure is closely linked to the question of **verificationism**: A theory is verificationist if all facts about the world can in princple be know.

It has been massively debated how one should consider and interpret gauge excess structure. For an attempt at defining and analyzing this structure using category theory see e.g. Weatherall (2015). Facing this presence of excess structure, one can take three different approaches[73]:

1. **Literalism**: We can accept that all the models related by a symmetry transformation represent different (but yet indistinguishable) physical situations. In this framework, we say that all the transformations are **active** transformations as opposed to **passive** transformations which would relate two different descriptions of the same physical situation.

2. **Reductionism** or **eliminativism**: We can attempt at rewriting the content of the theory such that everything is expressed in terms of a gauge invariant quantity to which one attributes the ontological status. As such, the redundancy disappear and can be treated as superfluous and unphysical.

3. **Sophistication**: We can accept that the presence of excess structure is necessary (or at least not problematic) but that all the symmetry related situations represent the same physical situations. This can be done by reformulating the theory (change its semantics) such that the symmetries act as isomorphisms[74] (Dewar 2019; Martens and Read 2020; Pooley 2006; Pooley 2013). About gauge sophistication see also Gomes (2022), Jacobs (2023) and Gomes (2026). In this framework, all transformations are understood as passive.

Furthermore, studying gauge theories and their interpretations allows to also investigate the validity of some much stronger metaphysical positions. These positions can be approached and argued for from different choices of interpretations and we will focus on them in the second part of this work regarding geometrical interpretations. Such metaphysical positions, as well as the different approaches to symmetry discussed previously, are largely inherited from the consequent literature on the physics of space-time, notably driven by the so called "hole argument" (Norton, Pooley, and Read 2023). As relevant metaphysical positions, we will consider here:

- **Scientific realism and substantivalism**: object described by physical theories are real.

[73] We consider here the three options proposed in Jacobs 2023.
[74] For definitions of isomorphisms see Appendix C.1.

- **Structural realism**: the ontological status of scientific theories should not be attributed to the object themselves but to the relations between them (for a review see (Ladyman 2023)). Such a position can be declined in various form. Notably one can take an ontological form, known as ontic structural realism (objects do not exist, only relations do) or an epistemic version, the epistemic structural realism (we cannot know about objects themselves, only their relations) (Ladyman 1998).

Other key metaphysical concepts are defined in App. C.2.

### 12.2 The "ABC" of interpretations

| | $A$ | $\mathcal{A}$ | B | C |
|---|---|---|---|---|
| point-like interaction | ✓ | ✓ | × | × |
| local action | ✓ | ✓ | × | ✓ |
| separability | × | ✓ | ✓ | × |
| measurability | × | × | ✓ | ✓ |
| Determinism | × | ✓ | ✓ | ✓ |
| Reduction | × | × | ✓ | ✓ |

Table 1: Updated table from Lyre (2004) for the four cases discussed here. The red checkmarks for the B-interpretations means that this is only true for the special case of classical electromagnetism. As discussed in the text, these checkmarks become crosses in the case of non-Abelian gauge theories.

Let us now investigate what could be the interpretations of gauge theories and their associated consequences. The question here is: if gauge theories factually describe some sort of reality, to which object should we attribute the ontological/physical status?

As suggested by Lyre (2004), the most obvious choices lie in the 'ABC interpretations'. A-interpretations would consider that the potential vector $\vec{A}$, or the gauge fields $A^{\alpha}_{\mu}$ is a physical object, B-interpretations attribute the ontological status to the magnetic field $\vec{B}$ (and the electric field $\vec{E}$) or the field strength $F^{\alpha}_{\mu\nu}$, while C interpretations consider that ontological status can only be attributed to phase factors/holonomies or Wilson loops. Following the terminology of Healey (2007) (chapter 4), these interpretations can also be understood as three broad categories dubbed respectively the "no-gauge property views", the " localized gauge potential property views" and the "non localized gauge potential property views". Let's have a detailed look at them starting with the $B$ interpretations which would be the natural point starting with the standard interpretation of classical electromagnetism.

### 12.3 The "no-gauge property view" or the B interpretation

Looking at how gauge fields appear in the classical theory of electromagnetism, one can think that they are pure mathematical tricks, simply used to derive the force fields in complicated situations. Indeed, one could legitimately assume that the vectors $\vec{E}$ and $\vec{B}$, or more generally the force field strength $F_{\mu\nu}$, are the only ontological quantities[75]. The $B$-interpretation have the quality to match our classical intuition about forces acting directly on objects. The electro-magnetic fields can directly be measured and are gauge invariant such that all the ambiguity of gauge transformations can be removed all at once. As such, this interpretation comes with a reduction of gauge symmetry (as defined in Sec. 12.1) in which all the information about gauge can be forgotten, hence the name "no gauge property view".

#### 12.3.1 The limitation from the formalism

One first hurdle to the B interpretation is that gauge fields seems to be required in order to formulate the theory in the formalism presented in Sec. 8.2. Indeed, no simple formulation of the gauge-matter coupling term can be formulated using only $\psi$ and the field strength. One could however argue that

[75]One could wonder if a $\vec{E}/\vec{B}$ interpretation can be considered as equivalent. As $\vec{E}$ and $\vec{B}$ are the components of $F_{\mu\nu}$ in a specific choice of spacetime frame, the answer would be yes.

this is simply a mathematical issue which has nothing to do with ontology. Our mathematical theory of the phenomenon would thus have to use abstract and "unreal" quantities to make predictions and the physically relevant quantities can always ultimately be recovered using Eq. 47.

#### 12.3.2 Lessons from The Aharonov-Bohm effect

As we discussed in Sec. 11, a major objection to the simple B-interpretation comes from The Aharonov-Bohm effect applied to electromagnetism. Simply put: $\vec{B}$ can influence $\psi$ in a region where $\vec{B} = \vec{0}$. As such, standing for the B-interpretation comes at some heavy cost regarding locality. By very definition, this influence of $\vec{B}$ on $\psi$ cannot be a point-like interaction. Without allowing for $\vec{A}$ to play the role of an intermediate propagator, the action of $\vec{B}$ on $\psi$ is also instantaneous and indirect, such that this interpretation cannot be considered as having local actions either. However, the whole process can be understood in terms of the disjoint subsystems $\psi$ and $\vec{B}$, such that this interpretation allows for separability. While this high cost for locality is commonly the primary motivation to reject the B interpretation, such brutal non locality is also found in phenomena as quantum entanglement and could thus be accepted as part of physical processes.

#### 12.3.3 Non Abelian theories and Wu and Yang's argument

Besides its implications for locality, the status of the B interpretation becomes severely problematic when applied to other gauge theories than electromagnetism, for which $G$ is a non-Abelian group. In that case $F^{\alpha}_{\mu\nu}$ is not a gauge invariant quantity, but transforms as $F^{\alpha}_{\mu\nu} X_\alpha \to (F^{\alpha}_{\mu\nu})' = h^{-1} F^{\alpha}_{\mu\nu} X_\alpha h$ under a gauge transformation. As such, B interpretations fail to be reductions in the non-Abelian case and we lose most of the main motivation for considering them: determinism, mesurability and separability. Indeed, contrarily to the electromagnetic case, $F^{\alpha}_{\mu\nu}$ cannot be uniquely determined at a time $t_0$ in the other gauge theories and its value later or earlier in time is determined by the time evolution only up to a gauge transformation. Moreover, we shouldn't be able to measure $F^{\alpha}_{\mu\nu}$ anymore in non abelian gauge theories, otherwise doing so would allow us to experimentally fix a preferred gauge. Finally, separability is also not satisfied as infinitely many $F^{\alpha}_{\mu\nu}$ can be considered to represent the effect of the field on $\psi$. As such, it becomes impossible to consider $F^{\alpha}_{\mu\nu}$ as one would interpret the electromagnetic field $\vec{E}$ and $\vec{B}$ as classical force fields. If we include non Abelian theories, the $B$ interpretations simply fail to satisfy any of the criterion we considered in Tab. 1.

As pointed out by Healey (2007), the situation is even more critical than that due to a powerful argument made by Wu and Yang (1975b). In their paper, the authors consider two different potentials $(A^{\alpha}_{\mu})_A$ and $(A^{\alpha}_{\mu})_B$ giving the same field strength $(F^{\alpha}_{\mu\nu})_A = (F^{\alpha}_{\mu\nu})_B$ while describing two physically different situations in the case of $G$ =SU(2). In the example considered, $(A^{\alpha}_{\mu})_A$ describes a source-less situation while $(A^{\alpha}_{\mu})_B$ describes a situation with sources, such that the two gauge fields cannot simply be related to one another by gauge transformations. The example is also considered over a simply connected region of space-time such that one cannot give any topological interpretation to the effect as it would be the case for the Aharonov-Bohm effect or any similar effect involving the electromagnetic force. As such, it might simply be impossible to argue in favor of a B interpretation for non Abelian theories.

Wu and Yang's example is rarely discussed in detail in the literature, but it provides, in our opinion, a much stronger case against the $B$-interpretations than Aharonov. It is presented in appendix H and we refer to the original paper for more details. The main point is that, in the case of non Abelian gauge theories, different physical situations can be described by the same field strength $F^{\alpha}_{\mu\nu}$, such that $F^{\alpha}_{\mu\nu}$ cannot be considered as the only ontologically relevant quantity of the theory.

As such, considering $F^{\alpha}_{\mu\nu}$ as the only ontologically relevant quantity and not $A^{\alpha}_{\mu}$ theories cannot be a viable path for non Abelian gauge theories, as different physical situations could not be distinguished by ignoring $A^{\alpha}_{\mu}$. We say that $F^{\alpha}_{\mu\nu}$ **under-determine** the theory.

#### 12.3.4 The rise and fall of the B interpretation

To summarize, due to the Aharonov-Bohm effect, $B$-interpretations for electromagnetism have to admit the existence of non local and non point-like actions of $B$ on $\psi$. With the addition of the gauge variance of $F^{\alpha}_{\mu\nu}$ for non abelian gauge groups as well as Wu and Yang's argument, one would have to further accept that B-interpretations are viable only for electromagnetism, while the strength tensor would acquire a different status for the other forces. This position is hardly sustainable and we thus have to accept that the gauge fields must encompass some physical effects that cannot be entirely expressed by their field strength tensor. One can thus be amused by imagining the incredible disappointment of Heaviside if he had lived long enough to see such a turn...

## 12.4 The local gauge potential properties and the A interpretation

### 12.4.1 A literal A interpretation

Recovering from the fall of the B-interpretation, the simplest interpretation one can give in this continuity is to associate some ontological properties to $A^{\alpha}_{\mu}$, corresponding to the scalar potential $V$ and the vector field $\vec{A}$ in electromagnetism. This interpretation is the so-called A interpretation presented in Lyre (2004). This choice is not too bad for our classical intuition, as this vector field present in each space-time point can couple locally to the matter fields to give rise to forces, much like $\vec{E}$ or $\vec{B}$ would. Additionally, we are not surprised anymore to see that $A^{\alpha}_{\mu}$ appears as so crucial within the formalism and we are seemingly able to dodge the bullet of the AB effect. Indeed, while $\vec{B} = 0$ outside of the solenoid, $\vec{A} \neq 0$, such that it can directly couple to $\psi$ in a point-like and local action. Actually, the case of the AB effect is clearly the most common arguments in favor of a A interpretation among physicists. However, while most physicist would today agree that the gauge fields are somehow the physical quantities at play in gauge theories, they rarely further dig into the limits of such a position.

A first obvious objection which can be raised is that $A^{\alpha}_{\mu}$ cannot be directly measured or quantified anyhow by experiments. As such, it is a bold choice to ascribe the ontological weight of a theory to something that can never be accessed directly experimentally. While $A^{\alpha}_{\mu}$ drives the whole theory, its effect is always indirect. If the opposite were true, we would be able to decide a preferred gauge experimentally and the theory would not truly be gauge invariant anymore.

More dramatically, the major issues with this interpretation comes from the obvious indetermination imposed on us by gauge invariance. Indeed, as any other gauge field related by a gauge transformation $A^{\alpha}_{\mu} \to (A^{\alpha}_{\mu})'$ describe the same situation, we are seemingly left with infinitely many different candidates to carry the ontological status of our theory. This interpretation forces us to take a literalist take on gauge invariance (as defined in Sec. 12.1), and one would consider that different gauge related $A^{\alpha}_{\mu}$ are different physical objects which correspond to different but indistinguishable physical situations. All gauge transformations are understood as active transformations. In its present form, neither experiment nor theory can distinguish between two gauge related situations, such that it is impossible to know the true value of the gauge field nor its evolution. As such, this interpretation is consequently both non separable and non deterministic.

The situation could be easier if any criterion could be used to break the gauge symmetry and select one specific $A^{\alpha}_{\mu}$ in the gauge orbit as the "one true gauge" (Aharonov and Anandan (1991) and Redhead (2002)). As such one could identify the single actually realized physical value for $A^{\alpha}_{\mu}$ which cannot be accessed directly by experiment. Such a preferred gauge fixing can be realized by introducing hidden ad-hock variable, or by arguing that one gauge choice is more natural or meaningful that the others (For a defense of the Coulomb gauge see Gomes and Butterfield (2021) and the unitary gauge see Wallace (2024)). It is however largely unclear if such a direction is anyhow meaningful and if we would be able to justify why so many different gauge potentials could describe the same observable situation. The problem is even worse in non-Abelian gauge theories, because of the so-called **Gribov ambiguity**, which state that gauge fixing is not enough to determine uniquely the state of the system (Gribov 1978).

### 12.4.2 The "new local gauge potential properties", geometric or $\mathcal{A}$ interpretations

One could try to avoid all these issues related to gauge invariance by identifying all the $A$ related to one another by gauge transformations as the same object $[A]$. Formally, one can define the equivalence relation $A \sim A'$ if $A$ and $A'$ are related by a gauge transformation of the second kind and build the equivalence class $[A] = \{(A_\mu^\alpha)' \in \mathfrak{A}$ such that $A_\mu^\alpha \sim (A_\mu^\alpha)'\}$, where $\mathfrak{A}$ is the set of all possible values of $A_\mu^\alpha$. Simply put: we could identify all elements sharing the same gauge orbit as being the same physical object. From there however, it is very unclear how to understand $[A]$ and its properties and we completely lose any possible understanding of gauge invariance. While such a proposition thus appears as a desperate and rather unjustified option to save the day, it turns out that it can be justified by exploiting the geometrical nature of gauge theories. Indeed, gauge theories share deep similarities with general relativity, so deep indeed, that it seems natural to propose a geometrical interpretation for them. As pointed out by many authors, the natural language to propose such an interpretation is given by the fiber bundle formalism. In such a framework, it is possible to argue that the different gauge fields $A_\mu^\alpha$ are the components in a different frame of a single geometrical object: a connection on a principal fiber bundle. Gauge transformation is thus interpreted as a frame transformation while the connection and its curvature bear the physical content of the theory. In such a framework, all gauge transformations are passive transformations and one can attempt at proposing a reduction of the theory in terms of the gauge invariant connection or alternatively a sophistication of the gauge symmetry (as defined in Sec. 12.1).

We will dub this geometrical interpretative program as "the geometric interpretations", the $\mathcal{A}$ interpretation (to distinguish them from the $A$ interpretations) or the "new local gauge potential properties views" following (Healey 2007). We use the plural form on purpose, as what should be exactly these interpretations and their philosophical consequences is still subject to debates.

As pointed out by Weatherall (2014), geometrical interpretations are often ignored or considered as the received view to be opposed in the philosophical literature. On the other hand, very few work proposed an in depth analysis of such interpretations. Leeds (1999) and Teller (2000) proposed some of the first articulations of a geometric interpretation, and Nounou (2003) considered it as a fourth explanation of the AB effect along with the A, B and C ones. The philosophical consequences of a geometrical framework for gauge theories were discussed in Guay (2004) and Catren (2008), while Weatherall (2014) and Gomes (2024a) recently proposed extensive overviews of the mathematical structure underlying such interpretations. Attempts at a metaphysical account of the fiber bundle formalism were given by Arntzenius (2012), Auyang (1995), and Maudlin (2007) and Jacobs (2023). According to Jacobs (2023), who proposes a metaphysical account of a sophisticated interpretation of the geometrical structure of gauge theories, such interpretation would satisfies our three criterion of locality (local actions, point-like interactions and separability) in a deterministic context. The next part of the present essay is dedicated at understanding and justifying these claims as well as investigating their possible limits.

### 12.5 The holonomy based view and the C interpretation

Another promising approach to interpret gauge theories, is to consider that the holonomies or Wilson loops given by Eq. 52, are the only physically meaningful quantities. Such interpretations were discussed notably by Healey (1997) and Lyre (2004) and wonderfully exposed in great details in Healey (2007). Such approaches propose a reduction of gauge symmetries by considering only the gauge invariant $\mathfrak{h}(\gamma)$. One of the strongest arguments to favor C interpretations over A or B interpretations is that $\mathfrak{h}(\gamma)$ provides a minimal account of gauge theories which does not under-determine (as $F$) nor over-determine (as $A_\mu^\alpha$) them in the sense that to a given value of $\mathfrak{h}(\gamma)$ corresponds one and only one physical situation and to a physical situation one can associate one and only one value of $\mathfrak{h}(\gamma)$. Holonomies follow a deterministic evolution and can be measured by reading off the change of the interference pattern into a phase difference in the case of the AB effect. While a C interpretation contains local action, it does not involve point-like interaction as $\mathfrak{h}(\gamma)$ acts on $\psi$ only on closed loop extended in space-time. In the case of the AB effect, the C interpretation is not separable as the property of the whole loop can be derived from the properties of its parts. Some criticism can be addressed to the C interpretation: the dynamics of the theory cannot be written in terms of loops and the rules for loop composition have to be supposed.

### 12.6 Other interpretative directions

While the "ABC" distinction provides a convenient and widely used classification of the main interpretative options for gauge theories, it does not exhaust the range of philosophical approaches that have been proposed in recent years. Several alternative programs attempt either to dissolve the interpretative difficulties associated with gauge symmetry or to reformulate the theory in a framework where the notion of gauge redundancy takes a different conceptual role. We briefly review some of these directions.

Building on ideas developed in Rovelli 2002, Rovelli 2014 and Rovelli 2020 proposes a relational interpretation of gauge. According to this view, gauge-dependent quantities such as $A$ or $\psi$ are not simply dispensable mathematical redundancies. Instead, they encode the ways in which a system may couple to other systems, providing "handles" for interaction. In a coupled theory, such quantities can combine across subsystems into gauge-invariant interaction terms, and it is these invariant combinations that are associated with observable effects. In other words: for individual systems, gauge dependent quantities are meaningless (and dispensable), just as "velocity" would be without any reference frame. But when systems couple, they become indispensable, and subsystems provide references for one another. See also the related discussions in Teh 2013 and Rivat 2023, as well as our discussion in Sec. 16.5.

Another strategy, discussed notably by Wallace (2014), questions the sharp ontological distinction between matter fields $\psi$ and gauge potentials $A_\mu$. In certain formulations of Abelian gauge theories, it is possible to rewrite the theory entirely in terms of gauge-invariant combinations of these quantities, suggesting that the apparent separation between matter and interaction fields may be partly conventional. From this perspective, gauge potentials do not represent independent physical entities but encode relational structure between matter degrees of freedom. While conceptually attractive, such constructions rely crucially on properties specific to Abelian theories. In non-Abelian gauge theories, the non-commutative structure of the gauge group obstructs a straightforward generalization, preventing a complete identification of matter and gauge variables. As a consequence, this approach does not currently provide a universal interpretative framework for gauge theories as a whole.

Another proposal consists in attempting to eliminate gauge redundancy altogether by reformulating gauge theories directly in terms of gauge-invariant quantities. A recent and systematic implementation of this idea is provided by the Dressing Field Method (DFM) (Berghofer, François, Friederich, et al. 2023; Berghofer, François, and Ravera 2025). The central idea is to construct composite fields — so-called *dressed fields* — obtained by combining the gauge potential and matter fields in such a way that the resulting variables are invariant under gauge transformations. In this framework, gauge symmetry is not interpreted as reflecting a physical indeterminacy but rather as an artifact of an enlarged mathematical description. By introducing suitable dressing fields, one can partially or completely remove gauge freedom and express the theory using quantities that transform trivially under the gauge group. Philosophically, this approach supports a form of reductionism: gauge symmetry is viewed as surplus structure that can, at least in principle, be eliminated without loss of empirical content. However, the construction is not always globally well-defined and may depend on specific features of the model under consideration, leaving open the question of whether gauge symmetry is truly eliminable in general.

A different line of analysis originates from the Hamiltonian formulation of gauge theories (Henneaux and Teitelboim 1992). In the constrained Hamiltonian dynamics program initiated by Dirac, gauge symmetries arise from the presence of first-class constraints in phase space. Gauge transformations are generated by these constraints and relate mathematically distinct states that correspond to the same physical configuration. The physical phase space is therefore obtained only after quotienting by gauge orbits. Within this perspective, the interpretative problem of gauge symmetry is closely tied to the identification of Dirac observables, namely quantities that commute with all constraints and are therefore gauge invariant. Gauge freedom is not introduced ad hoc but emerges as a structural feature of the dynamical formulation itself. Philosophically, this approach shifts the focus from fields to the structure of the space of states, suggesting that gauge redundancy reflects an over-complete description rather than genuine physical indeterminism. However, explicitly constructing Dirac observables is often technically difficult and typically leads to non-local quantities, thereby reintroducing tensions with locality.

More recently, several authors have proposed understanding gauge theories using the language of category theory and groupoids (see e.g. Catren (2023) and Weatherall (2015)). Instead of representing a physical theory as a set of models equipped with symmetries, one considers a category[76] whose objects are models and whose morphisms encode gauge transformations. In this framework, gauge-related configurations are naturally interpreted as isomorphic objects rather than distinct physical possibilities. Groupoids[77] provide a particularly natural mathematical setting for this idea, since they generalize groups by allowing transformations between different objects while retaining invertibility. Gauge symmetry is then interpreted as expressing equivalence relations internal to the representational structure of the theory. This approach supports a form of "sophisticated" interpretation of gauge symmetry, according to which excess structure is neither eliminated nor reified but understood through the relational structure between models. Although highly abstract, categorical approaches offer a unifying perspective connecting gauge theories with broader debates concerning theoretical equivalence, structural realism, and the role of mathematical representation in physics. Their philosophical implications remain an active area of research.

Taken together, these alternative directions illustrate that the interpretative landscape of gauge theories extends beyond the ABC taxonomy. Rather than providing mutually exclusive ontologies, many of these approaches attempt to reinterpret gauge symmetry itself — either by eliminating redundancy, redefining physical variables, or reformulating the mathematical framework in which equivalence between models is expressed. We shall come back to many of these ideas in Part III.

# Part III

# Geometrical interpretations of gauge theories: the fiber bundle ontologies

## 13 What is geometric about gauge theories?

### 13.1 Geometry on a surface

All we need to begin our journey into the geometric interpretation of gauge theories are the fundamental building blocks of differential geometry—most notably, the concept of curvature. We will avoid a purely formal or technical presentation here; for those seeking the rigorous machinery of fiber bundles and manifolds, please refer to Appendix D. Instead, we will lean heavily on visual intuition. For a complete presentation relying on such visualizations, see Needham 2021, Marsh 2014 and Faure 2022a; Faure 2022b.

One may imagine a general smooth surface $\mathcal{S}$. This could be anything from a flat sheet of paper to the curved leather of a football. Mathematically, $\mathcal{S}$ is called a **2-dimensional differentiable manifold**: around every point $x \in \mathcal{S}$, one can introduce local coordinates $(x^1, x^2)$ (for example by drawing a grid on a patch of the surface).

A **function** $f$ is the assignment of a number $f(x)$ at each point $x$ of $\mathcal{S}$ ($f : \mathcal{S} \to \mathbb{R}$). A **curve** $\gamma$ really corresponds to drawing a curved line on $\mathcal{S}$. Formally, it associates a segment of the real numbers to some specific set of points of the surface $\gamma : \mathbb{R} \to \mathcal{S}$.

Now imagine taking a pen and drawing a tiny arrow at every single point on this surface. More vividly, one could imagine gluing a toothpick to every point to create a giant mathematical hedgehog. This set of arrows could represent the velocity of a fluid or of many little bugs moving at each point of the surface. This collection of arrows is what is called a **vector field**, denoted by $v$ (in mathematics,

[76] A category is composed of objects, morphisms between these objects and a composition rule for the morphisms which is associative and has an identity. Relevant categories often consider objects of the same mathematical type and morphisms preserving some form of relevant structure (e.g. topological spaces and continuous maps), allowing to identify them.

[77] A groupoid is a set equipped with a unary inverse function and a binary operation which is a partial function (it is not defined for all pairs of the groupoid).

vectors are not written with arrows as in physics e.g. $\vec{v}$, and we will follow this convention here). At each point $x$, the arrow $v(x)$ lives in the **tangent plane**, noted $T_x\mathcal{S}$, the plane that best approximates the surface near that point.

To turn this intuition into a calculation, we can attach a local frame (or basis) $(e_1, e_2)$ to the tangent plane at each point. These basis vectors vary smoothly from point to point and span $T_x\mathcal{S}$, and they can be chosen properly such that $e_1$ is tangent to the direction of the coordinate line $x_1$ and $e_2$ is tangent to the direction of $x_2$[78]. Any vector field can then be decomposed as $v = v^a e_a = v^1 e_1 + v^2 e_2$, where the Einstein summation convention is used. Here: $v^1(x)$ and $v^2(x)$ are real-valued **functions** on $\mathcal{S}$ and $e_1(x)$ and $e_2(x)$ are vector fields forming a basis of the tangent space at each point, indices $a, b, c, \ldots$ take values $1, 2$. Thus, a vector field consists of two scalar functions once a local frame has been chosen. The geometric object $v$ itself is independent of this choice; only its components $v^a$ depend on the frame.

If we now wish to compare arrows at two different points $x_1$ and $x_2$, we immediately encounter a difficulty. The vectors $v(x_1) \in T_{x_1}\mathcal{S}$ and $v(x_2) \in T_{x_2}\mathcal{S}$ live in **different tangent planes**. There is no canonical way to subtract them. More generally one can ask, "I look at my vector field (arrow) $v(x)$ at a point $x$. How is the vector field changing, if I move in the direction $e_1$? Does it increase, does it turn?

To compare them, we must define what it means to move a vector from one point to another while keeping it "as rigid as possible." This procedure is called **parallel transport**. To formalise this idea, we define the **parallel transporter** $\Pi_{\gamma_{x_1 \to x_2}}$ which takes a vector $v$ and transport it as rigidly as possible along a curve $\gamma_{x_1 \to x_2}$ going from $x_1$ to $x_2$ to obtain a new vector at $x_2$, $v_{\text{par}}(x_2) = \Pi_{\gamma_{x_1 \to x_2}} v(x_1)$.

The parallel transporter $\Pi$ allows us to define properly the notion of a derivative of vectors on $\mathcal{S}$. Consider a point $x$ and a vector $w(x) = w^a e_a$ at that point. Consider a small shift of position from $x$ to $x\ \delta x$, where $\delta x$ is a small move of $x$ in the direction pointed by $w$.

We introduce $\nabla_w$ as:

$$\nabla_w v(x) = \lim_{\delta x \to 0} \frac{\Pi_{x+\delta x \to x}\, v(x + \delta x) - v(x)}{\delta x} \tag{53}$$

This derivative quantifies how the vector field $v(x)$ changes if you move on the surface in the direction pointed by another vector field $w(x)$ by taking the nearby value of the vector field located at $v(x + \delta x)$ in the direction pointed by $w$, parallel transport it back to $x$ through $\Pi_{x+\delta x \to x} v(x + \delta x)$ and compare it with the original value of the vector field.

The operator $\nabla$, taking a vector field $w(x)$ and acting on another field $v(x)$ is called an **affine connection**. $\nabla_{e_a}$, which we will note as $\nabla_a$, quantifies changes along the $e_a$ direction. It can be shown that such an operator gives back the standard derivative when acting on functions $\nabla_a f = \partial f / \partial x_a$ and that it obeys Leibniz rule, characteristic of derivatives $\nabla_a(fv) = f\nabla_a v + v\nabla_a f$. The **connection coefficients** $\Gamma^b{}_{ac}$ are defined as the covariant derivative of the basis vectors $e_c$ in the direction $e_a$:

$$\nabla_a e_c = \Gamma^b{}_{ac} e_b \tag{54}$$

and just quantifies how a basis is parallel transported over the surface. The properties of $\Gamma^b{}_{ac}$ encode what is meant by "parallel transport". To match the intuition of "holding as straight as possible on the surface", we ask parallel transport to preserve the lengths of the arrow and to be "torsion-free". This leaves a unique possibility for the coefficients, associated to the "Levi-Civita connection" (see Appendix D.3.1). Using the derivative rules as well as this definition of $\Gamma^b{}_{ac}$ on $\nabla_a(v^b e_b)$, we find the key expression:

$$\boxed{\nabla_a v = \left(\partial_a v^b + \Gamma^b{}_{ac} v^c\right) e_b} \tag{55}$$

With the $\nabla$ tool in hand, it will be possible to define the more intuitive notion of **curvature**. Curvature is defined as how much a vector turned from its initial direction when it is parallel transported back to its starting point around a loop. Indeed, imagine having a vector on the equator of a sphere. You can take a pen and hold it against a ball. Parallel transport the vector up to the north pole, for example

[78] This is the special choice noted $\partial_1$ and $\partial_2$, with the special property of vanishing commutator $[e_1, e_2] = 0$. Note that the frame $e_1$ and $e_2$ can be chosen in any other way, but for simplicity we will focus on such coordinate frames in this section.

by holding the pen rigidly against the surface and trying to induce no extra rotation. Arrived at the pole, make a 90° degree turn and go back to the equator. Make another 90° turn and go back to your original starting point. Your vector will have rotated compared to its original position, by an angle of 90°. The sphere is curved. Repeating the same kind of operation on a flat sheet of paper, or a cylinder will lead no rotation at all. These surfaces are flat. The angle the vector will make compared to its "past self" is named the **holonomy** of the connection.

The proper way to quantify these notions is through the so-called **Riemann curvature**, defined by its action on a vector $v$ as[79]:

$$\boxed{R_{ab}v = (\nabla_a\nabla_b - \nabla_b\nabla_a)v} \tag{56}$$

We see that $R$ quantifies the "commutation" of the covariant derivatives. It quantifies whether the vector, if parallel transported infinitesimally around a direction $a$ and then $b$, gives the same thing as if transported first along $b$ and then $a$. The Riemann curvature therefore measures the infinitesimal holonomy of the connection: it is the rotation per unit area generated by parallel transport around infinitesimal loops.

### 13.2 General relativity and the geometry of space-time

General relativity, our most successful theory of gravitation to date, is a natural generalization of the concepts discussed in the previous section. Instead of a two-dimensional surface, spacetime is modeled as a four-dimensional manifold $\mathcal{M}$, with the two extra dimensions accounting for the third dimension of space and one dimension of time. Just as a surface can be curved, spacetime can possess intrinsic curvature, described mathematically by the Riemann tensor. This curvature is determined locally by the distribution of energy and momentum. As put famously by J.A. Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve". The trajectory of particles in space-time is obtained from parallel transport ($\nabla_u u = 0$, with $u$ the four dimensional velocity vector). We say that they follow geodesics, the generalization of straight lines on a curved surface. The parallel transport is defined uniquely to be the so-called Levi-Civita connection (which is torsion free and preserve the "4 dimensional lengths" associated with the spacetime metric). In this sense, general relativity is considered as the archetypal geometric theory: gravitation is not a force acting within spacetime, but the manifestation of spacetime's geometry itself. Of course, the full richness of general relativity cannot be captured in a brief discussion; more detailed treatments are provided in Appendix B and the references therein, and debates over its fundamental interpretation about what is exactly meant by geometry remains active within the philosophy of physics community.

### 13.3 Analogy between geometry on a surface and gauge theories

After all these discussions, we are finally able to reach the most important point of our discussion. We will see that the simple example of geometry of the surface treated in Sec. 13.1 can be rewritten in a way that shows strong similarities with covariant electromagnetism. This simple example is the core of what is meant by "geometry of gauge theories". This example is freely inspired from Faure 2022a. Here again, we will skip most of the technical details, which can be found explicitly in Appendix G.1. Each tangent space in which vectors can live is a two dimensional vector space. As such, it is equivalent to the complex plane $\mathbb{C}$ (isomorphic as a vector-space). We can thus 'complexify' each vector space and treat it as the complex plane[80]. Recall that in the complex plane, rotations of a complex number by a number $\theta$ means multiplying it by $e^{\mathbb{i}\theta}$. As such, $e_2$ can be recovered by a rotation of $e^{\mathbb{i}\pi/2} = \mathbb{i}$ of $e_1$, that is $e_2 = \mathbb{i}e_1$. A general vector in the complex plane is equivalent to a complex number $v = v^1e_1 + v^2e_2 = v^1e_1 + v^2\mathbb{i}e_1 = \psi e_1$ with the complex function $\psi(x) = v^1(x) + \mathbb{i}v^2(x)$. With this

[79] In a natural frame where $[e_a, e_b] = 0$. See Appendix D.4.

[80] As detailed in Appendix G.1, this formally require the introduction of a complex structure.

new structure, the covariant derivative becomes[81]

$$\boxed{\nabla_a \psi = (\partial_a + \mathbb{i} A_a)\psi} \tag{57}$$

where $A_a = \Gamma^2_{a1}$, must be a real quantity if we enforce the "length preserving" condition of parallel transport. Curious! This looks just like the covariant derivative for electromagnetism! How about the curvature then? If we introduce the imaginary curvature $F_{ab} = \mathbb{i}R_{ab}$, Eq. 56 becomes:

$$\boxed{F_{ab} = \mathbb{i}R_{ab} = \partial_a A_b - \partial_b A_a} \tag{58}$$

Curious again! But there is more to come. What if we decided to change the frame in which the vector $v$ is expressed? That is, what if we change the direction of $e_1$? We could rotate arbitrarily the direction of $e_1$ randomly at different point of the surface. The proper way to do so is to do the rotation $e_1' = e^{\mathbb{i}\Lambda(x)} e_1(x)$, where $\Lambda(x)$ is an arbitrary angle of rotation at each point $x$ of the surface. This transformation should of course not change the geometrical quantity $v$, as it is just a change in the arbitrary frame we use to measure its components. In order to preserve $v$, the components $\psi$ should change accordingly in this new basis to be $\psi' = e^{-\mathbb{i}\Lambda(x)}\psi$. So $v = \psi' e_1' = \psi e^{-\mathbb{i}\Lambda(x)} e^{\mathbb{i}\Lambda(x)} e_1 = \psi e_1$. Here comes yet another coincidence! If we do such a transformation, $A_a$ happen to transform exactly As $A_a' = A_a + \partial_a \Lambda$. $F_{ab}$ remains unchanged. So to summarize, when we perform an arbitrary frame transformation on the surface, the quantities of interest transform as:

$$\begin{cases} \psi & \to \psi' = e^{-\mathbb{i}\Lambda(x)}\psi \\ A_a & \to A_a' = A_a + \partial_a \Lambda \end{cases} \tag{59}$$

which is exactly a gauge transformation (Eq. 44)!

Now imagine the specific case where the surface $\mathcal{S}$ is a cone. A cone is a very specific object, as it is flat everywhere, except at its apex, where it is curved. If a vector is parallel transported around a loop $\gamma$ containing the apex, it becomes rotated from its original self (holonomy) by an angle: $\varphi = -\oint_\gamma A_\mu \mathrm{d}x^\mu = -\int_\Sigma F_{\mu\nu}\, \mathrm{d}\Sigma$ which is analogous to the expression of the phase acquired by the wavefunction in the case of the Aharonov-Bohm effect! And that's not all. The two charge free Maxwell equations (Eq. 34) also come directly from mathematical properties of the curvature. More precisely it comes from the so-called Bianchi identity ( $\mathrm{d}_D R = \mathrm{d}_D^2 \Gamma = 0$, see Appendix E), which ensure that infinitesimal holonomies are consistent when glued together. This is the "deep geometrical meaning" we mentioned at the end of Sec. 4.2. The only missing part which do not seem to naturally flow as a geometrical consequence are the charged Maxwell equations.

Since both general relativity and gauge theories can be expressed in geometrical terms, it becomes difficult to refrain from asking: **is general relativity a gauge theory?** This question is at the heart of multiple debates. We would be tempted to answer positively to this question. The formulation of general relativity in the presence of a matter field ($\psi$) is exposed in Appendix B, making the similarities of the formalisms even more striking. However, some key disimilarities remains, ultimately leading to the fact that it is highly difficult to quantize general relativity, while it is possible to do so for gauge theories. The fundamental difference is that general relativity is the theory of geometry of space-time itself (just like the surface geometry) while gauge theories are about the geometry of internal spaces living "over" space-time: the internal spaces.

### 13.4 Fiber bundles and gauge theories

In the last section, we saw that **the geometry of vectors living on a surface can be rewritten in a similar language than a gauge theory of electromagnetism.** This is too good to be a coincidence! But is it just a nice way to think of it or can it provide us with a philosophically robust way to interpret gauge? The point of geometrical interpretations is exactly this one: the examples taken above can be somewhat taken literally as a faithful description of the gauge phenomena.

In this context, we would be tempted to provide the following interpretation:

[81]Simply use the definition for the covariant derivative of a vector $\nabla_a v = \nabla_a(\psi e_1) = \nabla_a \psi e_1 + \psi \nabla_a e_1$. $\psi(x)$ is a complex function $\nabla_a \psi = \partial_a \psi$ and $\nabla_a e_1 = \Gamma^1_{a1} e_1 + \Gamma^2_{a1} e_2 = (\Gamma^1_{a1} + \mathbb{i}\Gamma^2_{a1}) e_1$. Adding the condition that the parallel transport must conserve the length, we can show that $\Gamma^1_{a1} = 0$, hence leading to $\nabla_a e_1 = \mathbb{i}\Gamma^2_{a1}$. We then project on $e_1$ and introduce $A_a = \Gamma^2_{a1}$.

- Gauge invariance is simply the invariance of the arbitrary choice of frame above each surface used to express the physical quantities. For wavefunctions this would be the zero phases (the real axis of the complex plane).
- The potential field encodes how to parallel transport a frame from one space(-time) point to another. Its value depends on the choice of frame.
- Minimal coupling is simply the proper way to write a covariant derivative as a geometric necessity when using non trivial coordinates or in curved manifolds.
- The Aharonov-Bohm effect is explained by the presence of curvature inside the loop contained by the trajectory.

Now, is this a tenable position and what would this imply in terms of the concepts introduced in Part II? Let us see more clearly how we could generalize this example to arrive at a proper description of gauge theories. This formalism will furthermore allow to understand the similarities and the differences between general relativity and gauge theories, in a way that resembles our discussion comparing vector fields on surfaces and electromagnetism.

Space-time is modelled by a four dimensional manifold $M$ and not a two dimensional surface $\mathcal{S}$. Vectors in its tangent space can "point" in four orthogonal directions and it is thus four dimensional. The matter fields $\psi$ of gauge theories are complex objects (spinors) themselves contained in arrays of dimension 1, 2 and 3 to model electromagnetism, the weak force[82] and the strong force respectively, and it is clear that we will not be able to always identify the spaces in which the matter field lives with the tangent space of space-time, as we did in the analogy presented in the previous section. We will have to admit that matter fields can live in their own vector spaces, called **internal spaces**. Using the concept of fiber bundles, we can consider the existence of such an internal space "attached" at each point $x$ of space-time. The frames $e_a$ in which the vectors of the internal space can be decomposed are members of what is called a **principal bundle**, which contains all the possible frames over each point $x$, related to one another by transformations of the group $G$. The union of the internal space and their frames at each space-time point will be modeled by the concept of **associated vector bundle**. Gauge invariance, at least in what can be understood as a passive form, is simply the invariance of the geometrical objects under a frame transformation that is a change of point in the principal bundle. A connection can be defined on the principal bundle which translates how the frames changes when they are "dragged" along space-time. This connection is such that it preserve length in the internal space, associated to quantum probabilities. Gauge fields will be identified as the connection acting on the associated vector bundle where matter fields live. The curvature of this connection, will be identified with the gauge field strength. This curvature quantifies the rotation that a matter field would witness around a loop across the internal spaces over space-time, and gives rise to the fundamental interactions.

The proper mathematical formulation of the above text, in the context of fiber bundles is a rich and subtle topic. Thankfully, it is well known, and has been developed for around a century. For pedagogical introduction see Baez and Muniain 1994; Coquereaux 2002; Penrose 2004. For a technical introduction, see Appendix D.2 and the references therein. For a complete but dense review on the fiber bundle formalism in gauge theory see Marsh (2016) and for a presentation more oriented towards philosophy of physics see Weatherall (2014) and Gomes (2024a). A full technical presentation of fiber bundles is given in Appendix D and a formulation of gauge theories in the language of fiber bundles is presented in Appendix G.2.

Now, let's try to rephrase what was said before in a slightly more technical way, at least in order to introduce some of the key terminology for what will follow. We can say that a gauge theory can be described geometrically by saying that space–time is a smooth four–dimensional manifold $(M, g)$. At each point of $M$ one can "attach" another space. These spaces are called **fibers** over a point $x \in M$ and all the fibers together form a **fiber bundle**. The fiber bundle in which the matter fields $\psi$ live are complex linear vector spaces $V$. The states in these vector spaces represent the probability of presence

[82] Dimension 2 for left handed chiralities. Right-handed particles are singlets for the weak force. Weak force is particularly complicated because of its violation of the parity as well as the Higgs mechanism giving mass to bosons. We will not discuss it further here.

(wavefunction) for electromagnetism, and the probability for a particle (e.g. a quark) to be of one of the three colors: red, green or blue, for the strong force. In the case of the geometry of a surface, these space were real two dimensional tangent spaces. All possible choices of internal reference frames for such vector spaces are related by elements of a group $G$. If we select only orthonormal frames, the groups become U(1), SU(2) and SU(3) respectively for electromagnetism, the weak and the strong force. In the case of the surface geometry, the group was the group of rotations of orthonormal frames (preserving length and orientations) in 2D SO(2), which is equivalent to U(1) (hence our analogy). At each point, all possible frames are gathered into what is called a bundle $P$ over $M$, called the gauge bundle: above each space–time point $x$ sits a copy of the group $G$, representing all equivalent ways of choosing an internal frame. $P$ is also a fiber bundle, with fiber at each point being a copy of the group $G$. Such a bundle is called a **principal bundle**. Now, a group $G$ can admit different representations $\rho$ that is different ways to act on different vector spaces. This is crucial, as the representation will be linked to the charge of the particle, and explain why different fields will react differently to a same interaction (same principal bundle but different representations). Thus, the proper way to model the fiber bundle in which matter fields live is to consider a so called associated vector bundle $\mathcal{E} = V \times_\rho P$, where $V$ is a complex vector space carrying a representation $\rho$ of $G$; physically, $V$ is the internal space of the particle (its "charge space"), and a matter field $\Psi$ is simply a smooth assignment of a vector of $V$ at each point of a patch of space–time, expressed locally as $\Psi = [e(x), \psi(x)] = \psi^i(x)e_i(x)$ once a local frame $e(x)$ (a choice of gauge) has been selected. Such an assignement is called a **section** of the bundle. Note that such a choice cannot be done over all space-time at once if the topology of the bundle is not trivial. The central new ingredient of a gauge theory is a connection $\omega$ on the principal bundle, a Lie–algebra–valued one–form that specifies how internal frames are compared at neighboring points: it defines parallel transport and therefore tells us how to differentiate fields consistently from point to point. In a chosen frame over a region $e$, this connection appears as a Yang–Mills field $\omega_e = e^*\omega$, usually written in physics as a matrix–valued one–form $A = A^\alpha_\mu X_\alpha, \mathrm{d}x^\mu$, where $X_\alpha = \rho(\epsilon_\alpha)$ are the matrices representing the Lie algebra generators. This field acts on matter through the gauge covariant derivative $D_\mu\psi = \partial_\mu\psi + \overline{g}A_\mu\psi$, which modifies the ordinary derivative by adding a term proportional to the gauge coupling constant $g$, thereby encoding the interaction between matter and the gauge field. The failure of parallel transport to be path–independent is measured by the curvature $\Omega$ of $\omega$, which in a local gauge becomes the Yang–Mills field strength $\mathcal{F} = \mathrm{d}A + \overline{g}A \wedge A$; physically, $\mathcal{F}$ represents the force field (such as the electromagnetic or gluon field) itself. The Lagrangian of the theory can then be written geometrically as the sum of a pure gauge term $\frac{1}{2}\mathcal{F} \wedge \star\mathcal{F}$ and a matter term $\langle\Psi|(\not{D} - m)\Psi\rangle_V$, where $\not{D} = \gamma^\mu(\partial_\mu + \overline{g}A_\mu)$ is the Dirac operator, and the equations of motion follow from an extremalization of the action, yielding the Dirac equation $(\not{D} - m)\Psi = 0$ for matter and a generalized Maxwell equation $\star\mathrm{d}_D \star \mathcal{F} = \overline{g}\mathcal{J}$ for the gauge field (see the appendix for a definition of $\mathrm{d}_D$). Gauge invariance expresses the fact that physics does not depend on the arbitrary choice of internal frame: one may either view a gauge transformation as a passive change of local frame leaving geometric objects unchanged, or as an active bundle automorphism transforming simultaneously the matter components $\psi \to \rho(h)\psi$ and the gauge field $A \to hAh^{-1} - h^{-1}\mathrm{d}h$, in such a way that the covariant derivative and all the equations of motion remain invariant. In the Standard Model of particle physics, the full gauge group is U(1)×SU(2)×SU(3), acting on internal spaces such as $V = \mathbb{C}\otimes\mathbb{C}^2\otimes\mathbb{C}^3$, and different particles correspond to different representations (their "charges") of this same underlying principal bundle; in this sense, all fundamental interactions arise from a single geometric principle: the presence of a connection on a bundle encoding how geometrical degrees of freedom are parallel transported across space–time.

In summary, based on our previous illustrative example on a surface and its generalization allowing to reformulate the gauge theories in the language of fiber bundles, we are tempted to propose the following interpretative thesis a bit more formally than above:

- As Gomes (2024a) puts it: gauge theories are about the geometry of internal spaces.
- The matter fields $\psi$ used in physics are not geometrical objects but *components of geometrical objects* $\Psi = \psi^a e_a = (e, \psi)$. Confusion about gauge comes from ignoring this distinction.
- Connections $\omega$ encode the geometrical property of how any frame would change if transported

by the connection across space-time. The gauge fields are the values of the connection expressed in a specific space-time chart $U$ and considering a specific choice of frame $e$ for $P$ and $\mathcal{E}$ as $A = A^\alpha_\mu X_\alpha \otimes \, \mathrm{d}x^\mu$, which is a $\mathrm{End}(\mathcal{E})$ valued one form.

- The field strength tensors are the curvature of the connection $\omega$ also expressed in a specific chart and for a given choice of frame as $F = F^\alpha_{\mu\nu} X_\alpha \, \mathrm{d}x^\mu \otimes \, \mathrm{d}x^\nu$, which is a $\mathrm{End}(\mathcal{E})$ valued 2-form.

- Gauge invariance really is just invariance under a change of frame and contains nothing physical. The use of groups as $\mathrm{SU}(d)$ or $\mathrm{SO}(d)$ instead of $\mathrm{GL}(d)$ translate the restriction to a special group of frames, rendering explicit the conservation of a specific structure, namely the hermitian or the metric product.

- The gauge principle translates that frames can be changed arbitrarily at any point of space-time. As such the gauge principle is void of physical content and is not enough to create an interaction. Interactions can emerge only if $\omega$ is curved (at least at a given point of space-time like the case of the AB effect), corresponding to a non-zero strength tensor $F$ and if this strength tensor $F$ is given a kinetic term at the Lagrangian level.

Now while this interpretative framework is extremely rich and powerful, we shall explore its philosophical consequences and possible limitations.

# 14 Similarities and differences with general relativity/surface geometry

Indeed, one of the reasons to be genuinely excited about the fiber bundle formulation of gauge theories, is that it shows some significant and unexpected similarities between gauge theories and general relativity. Indeed, at the classical level, general relativity and gauge theories can be described using this same unifying mathematical formalism. These parallels have been used to shed some more light on the possible path and the difficulties to construct a quantum theory of gravity. As general relativity is often associated with some geometric interpretations, we expect that the same interpretations and the same debates can be applied to gauge theories. However, some differences between general relativity and gauge theories exist which might have implications for our interpretative attempts. Let us identify here these differences and their possible consequences. This will allow us to dispel possible mis-interpretation of gauge theories as geometry.

## 14.1 Tangent is not internal

By comparing Sec. 8.2 and Appendix B, we see that even without the fiber bundle formalism, gauge theories and general relativity exhibit important structural similarities. This becomes especially clear when matter fields are coupled to curved space-time using tetrad (or vierbein) fields. In this formulation, general relativity may be expressed using structures closely analogous to those of gauge theory, with a local Lorentz symmetry acting on orthonormal frames (see e.g. Hehl, Heyde, et al. 1976; Kibble 1961; Wallace 2015).

Within the fiber bundle formalism these similarities become more precise: both gauge theories and general relativity can be formulated as theories of matter fields represented by sections of associated vector bundles equipped with connections defining parallel transport and curvature. The natural question therefore arises whether — and if so where — the analogy between gravity and Yang–Mills theories ultimately breaks down.

Three differences are often invoked: the presence of a solder form, the role played by the metric, and the distinction between compact and non-compact gauge groups. We argue that none of these differences is fundamental. Rather, most of them reduce to a single geometric fact: the tangent space is not an internal space, because of its intrinsic relation to space-time itself.

In gauge theories, matter fields live in bundles associated with gauge principal bundles, which may be interpreted as abstract frame bundles of complex internal spaces. By contrast, in general relativity matter fields are described by bundles naturally associated with the frame bundle of space-time $LM$ (or, for spinor fields, with its spin lift). When restricted to orthonormal frames, this bundle corresponds to the collection of all local orthonormal bases of tangent spaces, related by local Lorentz transformations (rotations and boosts). Unlike internal frames, these frames are directly tied to directions in space-time through the geometry of the tangent bundle.

In several respects the tangent bundle is therefore a special bundle. Most immediately, it is intrinsically related to the base manifold itself, whereas internal bundles are not. Tangent vectors represent directions on space-time, while vectors in internal spaces do not correspond to directions on the manifold. Some authors have suggested that this special relation is encoded in the existence of the so-called solder form (see e.g. Anandan 1995; Healey 2007; Percacci 1984).

- **The solder form.** A solder form is a canonical $\mathbb{R}^4$-valued one-form $\theta \in \Omega^1(LM, \mathbb{R}^4)$, that is, a smooth map $\theta : T(LM) \to \mathbb{R}^4$ linear on each tangent space. For each frame $u \in LM_x$, it induces an isomorphism between the tangent space $T_xM$ and $\mathbb{R}^4$, allowing tangent vectors to be expressed in coordinates defined by that frame. Additional details are given in Appendix F. No canonical identification with the tangent bundle exist in ordinary gauge theories, since internal spaces are not naturally identified with tangent spaces of the base manifold[83]. Through the solder form, a principal-bundle connection acquires an interpretation as an affine connection on space-time, allowing torsion to be understood as a space-time tensor.

  Anandan 1995 argues that the existence of the solder form renders gravitational phases observable along open paths and therefore marks a fundamental disanalogy with gauge theories. However, as emphasized by Weatherall 2014, analogous identification maps arise in gauge theories when constructing associated bundles, so the solder form itself should not be regarded as decisive. The remaining difference instead reflects that general relativity is formulated using the tangent frame bundle $P = LM$, which is canonically tied to space-time.

  This relation also explains why phase differences may appear definable along open paths in gravity, whereas in gauge theories physically meaningful phase differences are typically associated with closed loops. Tangent vectors — like internal states — belong to distinct fibers and cannot be canonically compared at different points. Meaningful comparison arises only after parallel transport to the same point, leading to holonomy (Eq. 136). Apparent open-path comparisons arise because transported vectors can be compared with locally defined space-time structures such as frames or observers, rather than with their initial values. As discussed by Weatherall (2014) and Gomes (2022) (Chap. 3), this reflects the special relation between the tangent bundle and space-time geometry rather than any breakdown of gauge locality.

- **The metric.** Another apparent difference concerns the role of the metric tensor $g$ in general relativity. In gauge theories, internal spaces are similarly equipped with a Hermitian metric $\langle\cdot|\cdot\rangle$ defining an invariant inner product on internal states. Although these structures play analogous geometric roles, their interpretations differ: $g$ measures lengths and angles in space-time and is real valued, whereas $\langle\cdot|\cdot\rangle$ defines probability amplitudes for quantum states and is complex valued[84].

  As discussed in Sec. G.1, requiring $\Gamma^\lambda_{\mu\nu}$ to preserve $g$ ensures preservation of vector lengths, while requiring the gauge connection $A^\alpha_\mu$ to preserve the Hermitian metric leads to unitary gauge transformations and physically meaningful predictions such as the fact that $A^\alpha_\mu$ must be real. In standard Yang-Mills theory the Hermitian metric is treated as fixed, whereas $g$ is dynamical and varies across space-time. Nevertheless, generalized gauge theories may allow the internal metric itself to become dynamical, weakening this apparent disanalogy.

- **The gauge group and compactness.** One might instead argue that gravity should be understood as a gauge theory associated with the full frame bundle $LM$, whose structure group is $\mathrm{GL}(4, \mathbb{R})$, or even with diffeomorphisms of the manifold. The tetrad formulation corresponds

[83]In Sec. G.1 a complex structure allowed an identification between $T\mathcal{S}$ and $\mathbb{C}$; such identifications are not generally available for internal gauge spaces.

[84]It has been proposed by Sardanashvily (2005) and Sardanashvily (2011) that $g$ may be interpreted as a Higgs field; we do not pursue this idea here.

to a reduction of this bundle to the orthonormal frame bundle with structure group $\mathrm{SO}(1,3)$, determined locally by the metric $g$ (see e.g. Hehl, Heyde, et al. 1976; Kobayashi and Nomizu 1996). This mirrors Yang-Mills theory, where the presence of a Hermitian metric reduces the structure group from $\mathrm{GL}(n,\mathbb{C})$ to a unitary subgroup such as $\mathrm{U}(n)$ or $\mathrm{SU}(n)$.

A commonly emphasized distinction is that Yang-Mills theories typically involve compact gauge groups, whereas the Lorentz group is non-compact. Compactness plays an important technical role in quantum theory, ensuring positive-definite invariant inner products and well-behaved representations. However, this difference is not conceptually fundamental at the classical level: non-compact gauge groups can equally define consistent gauge theories, and the non-compactness of the Lorentz group ultimately reflects the indefinite signature of the space-time metric rather than a deeper structural separation between gravity and gauge theory.

More generally, one may formulate gauge theories using arbitrary internal frames, enlarging the structure group to $\mathrm{GL}(n,\mathbb{C})$ and allowing connections that do not preserve the Hermitian metric, in close analogy with metric-affine formulations of gravity (Hehl, McCrea, et al. 1995). Yet even this enlargement does not remove a key disanalogy: internal gauge transformations act vertically on bundle fibers, whereas diffeomorphisms act on the base manifold itself. The distinction between gravity and Yang–Mills theories therefore does not ultimately rest on orthonormality, compactness, or choice of gauge group, but on the special relation between the tangent bundle and space-time.

A related difference concerns dynamics: in standard general relativity the connection $\Gamma$ is uniquely fixed as the Levi–Civita connection, whereas the gauge connection $A$ is dynamical. Alternative formulations such as the Palatini approach treat $\Gamma$ as independent, further softening this distinction (Ferraris, Francaviglia, and Reina 1982; Palatini 1919). Because the solder form allows the $LM$ connection to be interpreted as a space-time affine connection, gravity admits equivalent reformulations in terms of curvature, torsion, or non-metricity (the "geometric trinity of gravity") (Beltrán Jiménez, Heisenberg, and Koivisto 2019), a reinterpretation without direct analogue in ordinary gauge theories, where curvature remains the primary geometric quantity.

Pursuing this discussion further, March and Weatherall (2024) argue that the deepest distinction lies in the geometric status of the bundles themselves. Although both theories are diffeomorphism covariant, gravitational fields live in natural bundles determined entirely by the structure of space-time $M$, whereas gauge-theoretic matter bundles depend on additional principal-bundle structure. Technically, $TM$ is a natural bundle, while $\mathcal{E}$ is more properly described as a gauge-natural bundle (Kolář, Slovák, and Michor 1993).

### 14.2 Could internal be tangent?

Higher dimension theories as the ones of Kaluza (1921) and Klein (1926a) and their generalization (including string theories), propose to understand the internal spaces as some additional spatial (compact) dimensions (see Sec. 7.4). However, there have not been so far any empirical argument in favor of such theories. If they happened to be verified however, they would clearly allow to understand gauge properties as literal space-time geometry within additional dimensions. For example, in standard Kaluza-Klein theory, gauge transformations as translations within the compact dimensions, charge as a quantized momentum within these dimensions.

Another crucial approach that is often neglected by philosophers of physics is the insight coming from the use of geometric algebra which denote the application of Clifford algebra to physics (or space-time algebra in a four dimensional context). Complete introductions to Clifford and geometric algebra can be found in Benn and Tucker (1987), Doran and Lasenby (2003), Doran (1994), and Dressel, Bliokh, and Nori (2015).

In Clifford algebra, matter fields can be naturally incorporated into an extended geometric structure associated with space-time: the **Clifford bundle**. Concretely, at each point $x$ of a space-time manifold $M$, the tangent space $T_xM$ is equipped with a metric $g$ and a basis $\{e_\mu\}$. From this, one can construct the **Clifford algebra** by defining the geometric product of vectors $u, v \in T_xM$ as $uv = g(u,v) + u \wedge v$, where $u \wedge v$ is the antisymmetric **exterior** (or wedge) product. This product combines the usual inner product with a fully antisymmetric outer product, producing an algebra that encodes both lengths and oriented planes.

The elements of the Clifford algebra can be decomposed on the basis $\{1, e_\mu, e_\mu e_\nu, e_\mu e_\nu e_\rho, e_0 e_1 e_2 e_3\}$, where $\mu < \nu < \rho$. If we rename the basis vectors as $\gamma_\mu$, we can identify them with the gamma matrices familiar from the Dirac equation [85] (Eq. 39). Importantly, these $\gamma_\mu$ should be understood **geometrically** as basis vectors of the Clifford algebra—they only appear as matrices when a representation is chosen.

The element $I = e_0 e_1 e_2 e_3$ represents a **unit oriented 4-volume** and behaves algebraically like the imaginary unit ($\mathbb{i}$). Hence, complex numbers arise naturally from the geometry of space-time itself, without introducing an external complex field.

A spinor $\psi$ can then be defined as an element of the **even subalgebra**, which consists of sums $\psi = S + B$ of complex scalars ($S = \alpha + I\beta$, $\alpha, \beta \in \mathbb{R}$) and bivectors ($B = \sum_{\mu<\nu} \alpha_{\mu\nu} \gamma_\mu \wedge \gamma_\nu$). This geometric perspective clarifies the structure and transformation properties of spinors, revealing their deep connection to space-time geometry.

For the purposes of gauge theories, this geometric viewpoint is especially enlightening. Since matter fields live in a "geometric extension" of space-time, operations that would normally be considered "internal" become geometric transformations within the Clifford algebra. Gauge transformations, in this context, can be understood analogously to rotations in this extended space, rather than as abstract operations on an internal space. Such an interpretation appear explicitly clear at least for the U(1) and the SU(2) transformations Hestenes 1986. Recent work on SU(3) gauge theories suggests that all fundamental interactions might be expressible in this geometric language (Lasenby 2024). In such a framework, both matter and gauge fields could be interpreted as intrinsic elements of an extended space-time, with gauge transformations being associated with specific types of geometric rotations.

A bit more formally, the geometric algebra framework suggests that gauge transformations can be represented as rotor transformations in specific bivector planes of the Clifford algebra. In that formal sense they resemble rotations. However, these transformations do not act on spacetime vectors via the sandwich map $v \mapsto Rv\tilde{R}$ (where $R$ is a rotor and $\tilde{R}$ its geometrical reverse). Rather, they act on spinors $\psi$, which are elements of the even subalgebra of the Clifford algebra.

Spacetime rotations and boosts act on spinors by left multiplication, $\psi \to R\psi$, and induce the corresponding transformation of spacetime vectors. By contrast, gauge transformations act by right multiplication, $\psi \to \psi R$. Such transformations do not rotate spacetime directions; instead, they modify the algebraic degrees of freedom encoded in the spinor while leaving appropriate bilinear spacetime observables invariant.

This suggests that gauge symmetry may be interpreted as reflecting intrinsic automorphism freedom of the local Clifford algebra structure associated with spacetime fields, rather than transformations in an independently existing internal space. Nevertheless, this interpretation must be handled with care, since the algebraic representation of gauge symmetry does not by itself eliminate the need for gauge connections or the standard dynamical structure of gauge theories.

### 14.3 The need for frames

Another apparent difference between GR and gauge theories is their relationship with frames and frame bundles. In particular, it might seem possible to write all of general relativity without invoking the frame bundle (LM), while a principal bundle is required in gauge theories. Gomes 2024a argues that this difference is only superficial and that we can think of parallel transport and connections without the need for frame bundles. Indeed once we have an associated bundle $\mathcal{E}$, we can build a vector bundle $E$ which fibers are vector spaces and forget everything about the frame bundle $P$. This point and its consequences are further discussed in Gomes 2024b.

[85] An analogous construction in three dimensions gives the Pauli matrices, which are themselves bivectors $\sigma_k = \gamma_k \gamma_0$ in a four dimensional context.

### 14.4 The gauge argument

As stated in Sec. 10, an analogue of the gauge argument can also be identified in the context of general relativity, and more generally in differential geometry.

Consider a vector field $v$ defined on a surface or on space-time. In a local frame $\{e_\mu(x)\}$ of the tangent bundle, it can be written as $v(x) = v^\mu(x)e_\mu(x)$. The geometric vector field $v$ itself is independent of the choice of frame, although its components and basis vectors change under a local change of frame

$$e_\mu \to e'_\mu = \Lambda_\mu^{\ \nu} e_\nu, \qquad v^\mu \to (v^\mu)' = (\Lambda^{-1})_\nu^{\ \mu} v^\nu. \tag{60}$$

Such transformations correspond to different local choices of basis in each tangent space and therefore represent alternative descriptions of the same geometric object rather than distinct physical configurations. When $\Lambda$ is constant over the manifold, this is a **global transformation**; when it depends on position, it defines a **local change of frame**.

Difficulties arise when one considers how the vector field varies across the manifold. A naive derivative of the components, $\partial_\nu v^\mu$, is covariant only under global transformations but not local ones ($\partial_\nu v^\mu = \partial_\nu (v^\mu)' = (\Lambda^{-1})_\lambda^{\ \mu} \partial_\nu v^\lambda$ only if $\partial_\mu \Lambda = 0$). As such, any equations of motion for $v$, as the geodesic equation, $v^\nu \partial_\nu v^\mu = 0$ would not be invariant (keep the same form) under a local change of frame. This is of course the equivalent of the free Dirac equation (Eq. 39). The reason is geometric: the partial derivative compares components defined in different tangent spaces without specifying how those spaces are to be related. Consequently, $\partial_\nu v^\mu$ does not define a tensorial object. The resolution is to introduce a connection, which specifies how vectors in neighbouring tangent spaces are compared. The covariant derivative of $v$ is defined as $\nabla_\nu v^\mu = \partial_\nu v^\mu + \Gamma^\mu{}_{\nu\rho} v^\rho$, where $\Gamma^\mu{}_{\nu\rho}$ are the connection coefficients in the chosen frame. This is the equivalent of the **minimal coupling rule** for gravity. The connection compensates precisely for the non–covariant behaviour of the partial derivative, only if it transform exactly as

$$(\Gamma_\mu)' = \Lambda \Gamma_\mu \Lambda^{-1} - (\partial_\mu \Lambda)\Lambda^{-1}, \tag{61}$$

which is the defining feature of a connection. This transformation law is structurally analogous to that of gauge potentials in Yang–Mills theories (8.2), highlighting the close mathematical relation between affine connections in geometry and gauge connections on principal bundles. In the above expression, we adopt matrix notation and therefore suppress the $\mathrm{End}(TM)$ indices of $\Gamma$, exactly as one typically suppresses internal indices in the representation of the generators $X_\alpha$ when writing the gauge connections.

Using $\nabla_\mu$ instead of $\partial_\mu$ ensure that equations remain the same regardless of the arbitrary local choice of frames, such as the geodesic equation $v^\nu \nabla_\nu v^\mu = 0$, which after a transformation remains $(v^\nu)' \nabla_\nu (v^\mu)' = 0$.

The introduction of $\Gamma$ therefore reflects the freedom to choose arbitrary local frames while maintaining a covariant description of physical quantities. In gauge theory, an analogous freedom appears as the possibility of redefining internal reference frames—for example, changing the local phase convention of a wavefunction in electromagnetism. In both contexts, the transformations express a redundancy in representation rather than a physical change of state.

Different types of fields couple differently to the connection depending on how they transform under changes of frame. Scalar fields transform according to the trivial representation and therefore satisfy $\nabla_\mu \phi = \partial_\mu \phi$, although the connection still determines the geometric structure within which derivatives are defined. Fields carrying non-trivial representations require additional structure. In particular, spinor fields transform under the spin group rather than directly under frame rotations and therefore require the introduction of a spin connection constructed from the metric connection and a choice of tetrad. This situation is closely analogous to the way different matter fields in gauge theory couple to gauge potentials according to their group representations, which are physically interpreted as charges.

Importantly, the existence of a connection does not presuppose curvature. Connections arise already from the requirement of comparing quantities defined in different tangent spaces and therefore exist even in flat geometry. Curvature appears only when parallel transport around closed loops becomes path-dependent; geometrically, it measures the non-integrability of parallel transport. The gauge argument thus motivates the introduction of connections as comparison structures, while the dynamical content of a theory determines whether curvature represents physical interaction or force.

In this thesis, we adopt the following interpretation of the gauge argument. At its core, the gauge argument expresses the requirement that the description of physical phenomena should not depend on arbitrary choices of reference frames used in their representation. In differential geometry, this principle manifests itself as the freedom to employ different coordinate systems or local frames on a manifold without altering the underlying geometric or physical content.

In general relativity this idea is closely related to the principle of general covariance, whose precise interpretative significance remains the subject of extensive debate in the philosophy of space-time (Earman 2006). We do not attempt to settle these debates here. Rather, we emphasize that general covariance plays a role analogous to the gauge principle insofar as both express invariance under changes of descriptive structure. The only limitation to this analogy, as before, is that tangent frame invariance/diffeomorphism symmetry acts on the space-time manifold itself, whereas internal gauge symmetries act on fibers attached to each space-time point (March and Weatherall 2024).

### 14.5 The equivalence principle

Both conceptually and historically, one of the empirical foundation of general relativity[86] is given by the Einstein equivalence principle (EEP). In its modern formulation, the EEP is itself defined as the composition of three fundamental assertions: i) local position invariance (LPI) ii) local Lorentz invariance (LLI) and iii) weak equivalence principle (WEP) (Will 2014; Will 2018). The consequence of the EEP is that space-time is locally flat or that it is always possible to find locally an inertial frame in which space-time is Minkowskian, with vanishing components of $\Gamma$. These frames are free-falling frames. Here again, the equivalence principle provides a strong empirical indication of the geometrical nature of gravity: it expresses the universality of free fall, according to which all test bodies follow identical trajectories in a gravitational field, and implies that freely falling frames define a local network of inertial frames across space-time.

As proposed by Lyre and Ahluwalia (2000), it is possible to define similarly a gauge equivalence principle by asking for a possible cancellation of the components of $A$ in each frame or through the universality of parallel transport in a unified internal space for all the fields of the standard model (Gomes 2024a). As in the previous cases discussed, partial analogues of these properties can be formulated in Yang-Mills theories, for instance through the local trivialization of gauge connections or the universality of parallel transport within a given representation, even though no exact counterpart of the equivalence principle exists due to the charge-dependent nature of gauge interactions.

The role played by the equivalence principle in gravity must be understood with some care when comparing it to gauge theories. The Einstein equivalence principle does not assert that all fields couple to gravity through identical dynamical equations, but rather that all forms of matter couple universally to the same space-time geometry. Fields of different spin transform under different representations of the local Lorentz group and therefore involve different realizations of the covariant derivative: scalar fields satisfy $\nabla_\mu \phi = \partial_\mu \phi$, spinor fields require a spin connection, and vector or tensor fields couple through the Levi-Civita connection. Nevertheless, these apparently different couplings are all induced by a single geometric structure—the metric and its associated connection — and therefore represent different representations of the same universal interaction (this need not be the case in generalisation of GR including e.g. torsion). In particular, even scalar fields, despite lacking an explicit connection term, remain coupled to gravity through the metric entering their kinetic terms and the invariant volume element $\sqrt{-g}$, so their dynamics depend on space-time curvature (as well as terms like $g_{\mu\nu}\partial^\nu \phi \partial^\mu \phi = g(\partial_\mu \phi, \partial_\nu \phi)$ in the Lagrangian). This situation contrasts fundamentally with Yang-Mills theories, where interactions depend on representation-dependent charges: neutral fields genuinely decouple from the gauge connection, while charged fields experience forces whose strength depends on their representation. The universality expressed by the equivalence principle therefore concerns the uniqueness of the geometric background rather than the uniformity of field equations, and it is precisely this universal coupling—absent in internal gauge theories—that supports the geometrical

[86] And of all theories known as "metric theories" of gravity. See Will (2014).

interpretation of gravity while limiting the analogy between gravitational and gauge connections. Ultimately however, this analogy resides again in the following argument: the tangent bundle is tied to space-time, in a way that internal spaces are not.

### 14.6 The gauge coupling constants and charges

In gauge theories, coupling constants and charges are related but different concepts quantifying how fields couple to an interaction. A coupling constant $\overline{g}$ is a parameter of the Lagrangian that sets the overall strength of the interaction. On the other hand, charges are invariant (like Casimir operators) associated with a representation of the gauge group under which a specific matter field transforms. Charges determine how a field couples to the corresponding gauge interaction and therefore characterize its sensitivity to the gauge connection. In general relativity, mass and spin may play a role partially analogous to charges in a representation-theoretic sense, since they label irreducible representations of the Poincaré group through its Casimir operators. Unlike gauge charges, however, these quantities do not lead to representation-dependent factors in the covariant derivative: in accordance with the equivalence principle, every field couple universally to gravity with the same intensity. Still, an analogy remains at the level of transformation properties: spin characterizes how a field transforms under local Lorentz transformations. In particular, a scalar (spin-0) field transforms trivially and is therefore invariant under local frame rotations and the spin connection does not act on it, in contrast with higher-spin fields. It might be considered as neutral, but, as discussed earlier it is still impacted by the metric at the Lagrangian level.

This raises the question of the role played by coupling constants and which one should be the one of gravity. In Yang-Mills theory, the gauge coupling constant can be absorbed into the definition of the connection, allowing the Lagrangian to be written with an overall normalization factor as $\mathcal{L} = -(16\pi\alpha_{\overline{g}})^{-1}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, where the gauge fine-structure constant is $\alpha_{\overline{g}} = \overline{g}^2/(4\pi)$. By comparison with the Einstein-Hilbert action $\mathcal{L} = (16\pi G)^{-1}R$, Newton's constant $G$ plays a role analogous to a gravitational fine-structure constant controlling the overall strength of the interaction.

However, the analogy breaks down on a crucial point: Newton's constant $G$ carries physical dimensions, whereas Yang-Mills fine-structure constants $\alpha_{\overline{g}}$ are dimensionless in four space-time dimensions. This dimensionality leads to qualitatively different quantum behaviors. In particular, perturbative quantum gravity based on the Einstein-Hilbert action is non-renormalizable, in close analogy with the historical role of the dimensionful Fermi coupling constant in the original theory of weak interactions (Fermi 1934).

The dimensional nature of the gravitational coupling constant therefore represents a major structural difference between general relativity and Yang-Mills gauge theories[87]. Once again, such a difference can be explained in the tangent vs internal context:

In Yang–Mills theory the interaction enters through the covariant derivative $D = \mathrm{d} + \overline{g}A$, where $A$ is a connection acting on internal fibers that carry no intrinsic length scale; hence in four dimensions one may take $[A] = L^{-1}$ so that $[\overline{g}] = L^0$ (dimensionless). In gravity the dynamical field is the space-time metric $g_{\mu\nu}$, which defines physical lengths, and the action contains the curvature scalar $R \sim L^{-2}$; requiring a dimensionless action therefore introduces a coupling $G$ with $[G] \sim L^2$. The dimensional character of $G$ thus reflects that gravity makes space-time geometry itself dynamical, rather than describing interactions within an independent internal space.

### 14.7 The Lagrangian and the dynamics

While both theories share common conceptual and mathematical tools, the perhaps most important and irreducible difference between general relativity and Yang-Mills theories is the difference between their Lagrangians and hence their equations of motions. To put things simply: "Both equations may be understood to relate the curvature of a principal bundle to properties of matter, but the relationships are simply not the same" (p.26) Weatherall 2014. General relativity's Lagrangian is linear with the

[87]In the comparison of gravity with other forces, it is sometimes relevant to introduce the dimensionless gravitational fine structure constant $\alpha_G = GE^2/(\hbar c^5)$, where $E$ is a characteristic energy scale. It is common to use the proton mass energy for such scale, $E = m_p c^2$, such that $\alpha_G = Gm_p^2/(\hbar c)$. See Silk (1977).

curvature, while Yang-Mills is quadratic. A similar Lagrangian ($\mathcal{L} = -F_{\mu\nu}^{IJ} F_{IJ}^{\mu\nu}/4$ where $F$ is the curvature of the spin connection) can be proposed for gravity, but clearly do not lead to the same phenomenology, and hence cannot be a candidate for a theory of gravity (Alexander and Manton 2023). However, formalism including this term with other terms, known as quadratic gravity, display important properties, as being easier to quantize than general relativity (perturbatively renormalizable) (Salvio 2018). However, they also predict the existence of new particles (ghosts) which shed doubt on its possibility to be empirically viable[88].

In this context, one can only hope that beyond standard model theories both for particles and gravity could produce deeper similarities at the lagrangian level.

### 14.8 Aharonov-Bohm effect

An instructive comparison between gauge theories and gravity is provided by analogues of the Aharonov–Bohm effect in space-time geometry (see Gomes 2025b). In gauge theory, the Aharonov–Bohm effect shows that the gauge potential may have observable physical consequences even in regions where the local field strength vanishes. The measurable quantity is not the local value of the connection itself, but the holonomy associated with parallel transport along a closed path.

A closely related phenomenon exists in space-time physics. When a vector is parallel transported along a curve on a curved manifold, its orientation may change due to the presence of a connection. However, this change cannot be quantified intrinsically at a different point of the manifold, since tangent spaces at distinct points are independent vector spaces. A meaningful comparison becomes possible only once the transported vector is brought back to its original point, allowing both vectors to be compared within the same tangent space. The resulting transformation is precisely characterized by the holonomy of the space-time connection. In other word, when brought back to istelf around a loop $\gamma$, a vector is transformed as $(v^\mu)' = \mathfrak{h}^\mu{}_\nu(\gamma) v^\nu$, with the holonomy $\mathfrak{h}^\mu{}_\nu(\gamma) = \mathcal{P} \exp\left(-\oint_\gamma \Gamma^\mu{}_{\nu\rho}(x)\, dx^\rho\right)$ (to be compared with Eq. 52).

This situation is entirely analogous to that encountered for internal vectors in gauge theories. Internal states belonging to different fibers cannot be directly compared at distinct space-time points; only parallel transport around a closed loop produces an unambiguous, gauge-invariant observable. In both cases, physical effects arise from global properties of the connection rather than from locally measurable quantities alone.

Gravitational Aharonov–Bohm effects therefore illustrate that, despite their different physical interpretations, gauge theories and general relativity share a common geometric feature: observable quantities may depend on the global structure of parallel transport encoded in the connection. At the same time, the comparison again highlights the special role of gravity, since the connection governing holonomy acts on the tangent bundle itself and therefore directly affects space-time geometry rather than an independent internal space.

### 14.9 On diffeomorphisms, locality and constraints

Some commonly invoked differences between Yang-Mills theories and general relativity emerges when one considers the role of diffeomorphism invariance, the notion of locality, and the structure of the Hamiltonian constraints. At the level of the Lagrangian formulation, both theories can be written in a manifestly covariant form and are therefore invariant under diffeomorphisms of the space-time manifold. In Yang–Mills theory this invariance expresses the coordinate-independence of tensorial equations written on a fixed background space-time. The gauge symmetry of the theory, however, acts independently of this coordinate freedom: it corresponds to transformations of the internal fibers of a principal bundle and leaves the points of the space-time manifold unchanged. In general relativity the

[88] Other formulations provide bridges between Yang–Mills and General Relativity Lagrangians. In the MacDowell–Mansouri formulation MacDowell and Mansouri 1977, gravity is written as a curvature-squared gauge theory for a connection with gauge group SO(4,1) or SO(3,2), the de Sitter or anti–de Sitter groups. Although spacetime remains four-dimensional, the connection takes values in the Lie algebra of these five-dimensional symmetry groups. After a symmetry breaking that reduces the gauge symmetry to SO(3,1), the action reproduces the Einstein–Hilbert action (with cosmological constant). For a pedagogical discussion see Wise 2010.

situation is different because the space-time metric itself is dynamical. Diffeomorphisms therefore act directly on the geometric field that determines distances, causal structure and parallel transport. As a result, diffeomorphism invariance is not merely a passive coordinate symmetry but becomes a genuine gauge symmetry of the gravitational dynamics.

This distinction has important consequences for the notion of observable quantities. In Yang–Mills theory, gauge transformations act only on internal degrees of freedom, so space-time points retain an invariant meaning. One can therefore construct gauge-invariant local observables defined at a point of space-time, such as local composite operators built from the field strength, for instance $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$, as well as gauge-invariant Wilson loops defined along closed curves. In general relativity, by contrast, diffeomorphisms move the points of the manifold themselves. A scalar quantity such as $R(x)$ or $g_{\mu\nu}(x)$ evaluated "at the point $x$" is therefore not invariant under diffeomorphisms, since the identification of points across configurations is not physically meaningful. This has led to the well-known claim that general relativity does not admit nontrivial local diffeomorphism-invariant observables constructed solely from the metric and finitely many of its derivatives (Torre 1993). Physical observables must instead be relational, defined through relations between dynamical fields or through quantities associated with extended structures such as holonomies or asymptotic charges (this is also closely linked to the "point-coincidence" argument). The restriction on locality is therefore not a consequence of gauge symmetry in general, but specifically of the fact that the symmetry acts on the base manifold itself.

These conceptual differences become especially transparent in the Hamiltonian formulation. The starting point is the Legendre transform, which replaces the variables $(q_i, \dot{q}_i)$ of Lagrangian mechanics by the phase-space variables $(q_i, p_i)$ of Hamiltonian mechanics with $p_i = \partial\mathcal{L}/\partial\dot{q}_i$. One then asks whether this relation can be inverted to express the velocities $\dot{q}_i$ uniquely in terms of $q_i$ and $p_i$[89] When this inversion fails, the Legendre transform does not fill all of phase space: it reaches only a smaller subset. As a result, the variables $(q_i, p_i)$ are not all independent but must satisfy equations of the form $\phi_a(q_i, p_i) = 0$ called constraints. The dynamics is then restricted to the surface of phase space defined by these constraints[90]. A simple way this can happen is when some velocity variables do not appear in the Lagrangian.

In the canonical formulation of Yang–Mills theory, the basic phase-space variables are the spatial components of the gauge potential $A_i^\alpha$ and their conjugate momenta[91] $E^{i\alpha}$, which correspond to the generalized electric fields. Because the time component[92] $A_0^\alpha$ appears in the Lagrangian without a time derivative, its conjugate momentum vanishes. This means that $A_0^\alpha$ can be understood as a Lagrange multiplier associated with a constraint. Extremizing the action with respect to it, one obtains Gauss' law, $(D_i E^i)^\alpha(x) + J^{0\alpha}(x) \approx 0$. This constraint generates internal gauge transformations[93] on the canonical variables. Configurations related by gauge transformations are understood as physically equivalent and physical states are therefore not individual points of phase space, but gauge-equivalence classes on the constraint surface. Time evolution itself, however, is generated by a genuine Hamiltonian defined on a fixed background spacetime. Gauge symmetry therefore restricts the allowed states but does not eliminate the notion of physical time evolution.

In general relativity the canonical structure is considerably richer. When the space-time metric is decomposed into spatial slices using the ADM formalism, the dynamical variables are the spatial metric $q_{ij}$ and its conjugate momentum $\pi^{ij}$. The lapse function $N$ and shift vector $N^i$ appear without time derivatives and therefore enforce constraints. The resulting Hamiltonian takes the form $H = \int d^3x \left(N\mathcal{H} + N^i\mathcal{H}_i\right)$, where $\mathcal{H}$ is the Hamiltonian constraint and $\mathcal{H}_i$ are the momentum constraints generating spatial diffeomorphisms (Arnowitt, Deser, and Misner 2008; Henneaux and Teitelboim 1992). In this formulation the Hamiltonian is itself a combination of constraints (and thus null), reflecting the fact that the freedom to choose how space-time is foliated corresponds to gauge

[89] This is possible when the Hessian $\partial^2\mathcal{L}/\partial\dot{q}_i\partial\dot{q}_j$ is invertible.

[90] As such, they appear coupled to a Lagrange multipliers in the action.

[91] $E^{i\alpha}$ is identified with the momentum conjugate to $A_i^\alpha$, $E^{i\alpha} = \frac{\partial\mathcal{L}}{\partial\dot{A}_i^\alpha}$, up to sign and index conventions.

[92] That is, $V$ for electromagnetism.

[93] Writing $\mathcal{G}^\alpha(x) = (D_i E^i)^\alpha(x) + J^{0\alpha}(x)$, and defining the smeared constraint $G[\lambda] = \int d^3x\, \lambda_\alpha(x)\mathcal{G}^\alpha(x)$, saying that Gauss's law generates gauge transformations means that $\delta_\lambda f = \{f, G[\lambda]\}$, just as $H$ generates time evolution and $p$ spatial translations. In particular, $\delta_\lambda A_i^\alpha = \{A_i^\alpha, G[\lambda]\} = (D_i\lambda)^\alpha$. In the abelian case of electromagnetism, this reduces to $\delta_\lambda A_i = \partial_i\lambda$..

redundancy. Consequently, the distinction between gauge transformations and dynamical evolution becomes subtle, giving rise to the well-known "problem of time" in canonical gravity.

The algebra satisfied by the constraints also reflects the geometric nature of the theory. In Yang–Mills theory the Gauss constraints reproduce the Lie algebra of the gauge group, with constant structure coefficients determined by the group structure constants. In general relativity, however, the Hamiltonian and momentum constraints satisfy the hypersurface-deformation algebra, whose coefficients depend on the dynamical spatial metric. The algebra therefore involves structure functions rather than fixed structure constants, reflecting the fact that the symmetry transformations correspond to deformations of hypersurfaces embedded in a dynamical geometry.

Taken together, these features illustrate that the differences between Yang–Mills theories and general relativity are not simply technical but stem from a deeper geometric distinction. In Yang–Mills theory, gauge transformations act vertically on the fibers of an internal bundle, leaving the space-time manifold itself unaffected. In general relativity, by contrast, the relevant gauge symmetry is generated by diffeomorphisms that act on the base manifold and therefore on the space-time geometry itself. Because locality is defined with respect to the base manifold, this difference has profound consequences for the structure of observables, the interpretation of dynamics, and the constraint algebra of the theory. The deep conceptual distinction between gauge theories and gravity can therefore be traced to the fact that Yang–Mills symmetries act on internal spaces attached to space-time, whereas the fundamental symmetry of general relativity acts directly on space-time itself.

In conclusion: Although both Yang–Mills theory and general relativity are diffeomorphism covariant, the role played by diffeomorphisms differs profoundly in the two theories. In Yang–Mills theory they primarily express coordinate invariance of fields defined on a fixed background spacetime, whereas in general relativity they act on the dynamical geometry itself. As a consequence, diffeomorphism symmetry becomes a gauge symmetry of the dynamics in GR, generating the Hamiltonian and momentum constraints and strongly restricting the construction of local observables. In the tetrad formulation presented in Appendix B, gravity possesses both local Lorentz symmetry and diffeomorphism invariance. The former resembles the internal gauge symmetry of Yang–Mills theories and is responsible for the appearance of a spin connection analogous to a gauge potential. The latter, however, acts on the space-time manifold itself because the tetrad identifies the internal Lorentz space with the tangent bundle. As a consequence, the gauge symmetry of gravity extends to transformations of spacetime geometry, leading to the distinctive constraint structure and the absence of strictly local diffeomorphism-invariant observables.

### 14.10 Summary: gravity and gauge share interpretative tension on the role of geometry

The preceding discussion suggests that the differences between Yang–Mills theory and general relativity can ultimately be traced back to two structural features. First, general relativity concerns the geometry of space-time itself, whereas Yang–Mills theories describe the geometry of internal spaces independent of space-time directions. Second, the two theories possess different dynamical laws relating geometry and matter fields: the gravitational action is linear in curvature, while Yang–Mills dynamics is quadratic.

Many apparent disanalogies discussed above — including the role of the solder form, the universality of gravitational coupling, and the dimensional character of Newton's constant — can be understood as consequences of these two underlying distinctions. These differences persist so long as no empirically supported theory or interpretation succeeds in unifying space-time and internal geometry or in modifying the dynamical relation between curvature and matter.

Nevertheless, we shall argue that most philosophical and conceptual issues arising in the interpretation of space-time geometry extend naturally to gauge theories[94], since both frameworks ultimately rely on the same geometric notions of locality, parallel transport, and redundancy of description.

[94] Of course some philosophical problems are specific to general relativity (problem of time, scarcity of local diffeo-invariant observables, gravitational energy localization ...) and some to Yang-Mills theory (confinement and the physical status of nonperturbative gauge-invariant operators, Higgs mechanism and gauge symmetry breaking...). We simply state here that they share enough geometrical structure (connections, curvature, holonomy, locality, representational redundancy...) to face numerous common interpretative problems.

As a general guiding principle for this part: **we should seek for an interpretation of matter fields, gauge invariance and gauge fields, that matches an interpretation we could do of the velocities, frame/diffeomorphism invariance and vector connections respectively, forgetting completely about the fact that velocities point in a direction of space-time (association possible only once a solder form is given) and about the details of the dynamics**. Once such an interpretation is found, we can see how it compares between general relativity and gauge theories.

## 15 Beyond ABC: fiber bundle substantivalism

From the previous section, it appears that, beyond the identified differences, the structural similarities between general relativity and gauge theories are sufficiently striking to warrant serious interpretative attention. In particular, the geometric reformulation of gauge theories presented previously suggests that their interpretation cannot be cleanly separated from the broader interpretative issues surrounding general relativity itself. If both theories admit a common differential-geometric formulation, then questions about the ontology of geometric structure arise simultaneously in both contexts.

The most direct — and arguably most enthusiastic — response to the reformulation of Sec. 13.4 is to take the geometric framework at face value and regard the relevant fiber bundles as genuinely existing structures, understood as abstract spaces systematically attached to space-time. On this view, gauge theories describe the motion of matter fields within geometrically structured bundles, and gauge interactions arise literally from geometric properties such as connections and curvature. Following Arntzenius 2012, we refer to this position as **fiber bundle substantivalism**, extending the traditional doctrine of space-time substantivalism[95], according to which space-time itself exists as an entity independent of the material contents of the universe.

This position possesses several notable interpretative virtues:

- **Unification**: gravitational and gauge interactions admit a common geometric interpretation, both being encoded in connection and curvature structures.
- **Explanatory depth**: interactions are explained geometrically, emerging from curvature rather than being introduced through apparently ad hoc interaction terms.
- **Local realism**: interactions are locally encoded through infinitesimal geometric structure, with parallel transport providing a natural account of how physical effects accumulate along space-time trajectories. We might thus hope that it will allow for a deterministic, local and separable interpretation, allowing for point-wise interactions (See. II).

The central difficulty, however, lies in distinguishing genuine physical structure from mathematical redundancy within the rich fiber-bundle formalism. As emphasized by Guttmann and Lyre 2000, gauge theories contain substantial representational freedom, and any substantivalist interpretation must explain if and why only certain geometric features — typically gauge-invariant ones — should be regarded as physically real.

We emphasize that similar questions already arise in the context of general relativity when considering the tangent bundle and its related structures. There one may work equivalently with the tangent bundle itself, the frame bundle, or the orthonormal frame bundle, together with the associated Levi–Civita connection. These mathematical constructions are not identical, but they provide closely related representations of the same underlying local geometric structure. In this case the interpretative problem may appear less severe, since these different bundles ultimately encode the familiar idea that, at each point of space-time, physical quantities are naturally matching our intuition inherited from surface geometry of a "flat sheet space" tangent at each point of space-time (which formally are linear vector spaces equipped with a local Lorentzian metric). Thus, we might be tempted to follow the same path for gauge theory and collapse the difference between all these representations. Whether such a collapse of representational distinctions is justified in the gauge case, however, remains precisely what is at issue.

[95]Nomenclature which was introduced by Sklar 1974 according to Stachel 2014.

According to space-time substantivalism, even in the absence of all matter and fields, space-time would persist as an independently existing entity. Space and time are therefore not merely relational structures but possess an ontological status of their own. This view echoes Newton's notion of absolute space underlying classical mechanics and recalls the well-known objections raised by Leibniz, who defended a relational conception of space (see the review in Pooley (2013)). Fiber bundle substantivalism extends this commitment further: not only does space-time exist independently, but so do the geometric structures attached to it. In particular, tangent spaces — and more generally fibers — are taken to exist even in the absence of vectors or fields occupying them; the spaces in which velocities, charges, or internal degrees of freedom may reside remain part of the ontology regardless of their occupation. Furthermore, such a position may imply some kind of realism about the entities living in the fibers. To keep the analogy with the tangent space, such a position, would claim realism about velocity vectors, as actual realised geometrical objects.

As will be discussed in the next section, a naive version of fiber bundle substantivalism faces substantial philosophical and physical challenges. Nevertheless, developed contemporary defenses of this position can be attempted, with a large amount of care.

# 16 Challenges for a substantivalism of fiber bundles

## 16.1 Fiber bundles everywhere!

One of the strongest appeal to a geometrical interpretation of gauge theories is based on "esthetic criterions". As discussed in Batterman (2003) in a work entitled "Falling Cats, Parallel Parking, and Polarized Light", the framework of fiber bundles allows to treat multiple unrelated problems. Even more surprisingly, it is possible to build simple toy models of the financial market using fiber bundles which have exactly the same structure as electromagnetism, including Maxwell equations Ilinski, Kalinin, and Stepanenko (1997), Maldacena (2014), Schwichtenberg (2019a), Schwichtenberg (2019b), and Young (1999). Such approach is actually deeply enlighting to understand gauge theories. In such a framework, the base space $M$ could be the planet earth as a manifold. Fiber at each point represent the prices of a commodity at each country. Gauge invariance is the local arbitrary choice of money unit. The connection represent exchange rates between different money units. Curvature is the fact that, making a loop between four countries, one can win or loose money (arbitrage opportunities).

Now, while this analogy is useful we can doubt that there truly exist some "money space" such that money currents are a consequence of their curvature as a geometrical property, in the intuitive sense of curvature. However, the fact that such a wide range of phenomena can be represented and understood geometrically using the fiber bundle formalism might reveal that this framework encodes some deep properties of systems. See our following discussion in Sec. 18.4.

Furthermore, such analogies may place additional pressure on ontologies based solely on holonomies and loops (type C interpretations in Sec. II). Indeed, in the financial model, the most relevant quantities are the gains or losses that can be realized by traversing a loop back to one's original country (that is, arbitrage opportunities). However, it seems clear that concluding from this that only loops and holonomies are "physical", denying the existence of some form of local money transfers between countries, would be an overinterpretation of the formalism.

## 16.2 The parsimony argument and the ontological inflation

For Lyre (2004), C interpretations are more favorable as they suppose less "surplus structure". They are minimal and only ask for what is needed to recover the physical observable. One would then "fall all to easy into a mysterious Platonism". This is a similar argument than one would use to favor e.g. Copenhagen interpretations over De Broglie-Bohm pilot wave theories. As such, according to Lyre, Wilson loops completely and minimally describe the effect of a gauge theory. Gauge transformations can be interpreted as the changes of connections which leave holonomies invariant (vertical bundle automorphisms). In its most naive formulation, fiber bundle substantivalism appears to commit one to the existence of a vast amount of unobservable structure. Such an ontological proliferation raises concerns of theoretical parsimony, and considerations in the spirit of Ockham's razor may therefore be taken to count against this interpretation, favoring accounts that avoid positing surplus structure beyond what is required to capture empirical content.

### 16.3 The hole argument

The hole argument is widely regarded as one of the most powerful challenges to space-time substantivalism. Originally formulated in the context of general relativity, it aims to show that attributing independent ontological reality to space-time points leads to an unacceptable form of indeterminism.

This line of reasoning can be naturally extended to gauge theories, where it has been developed as an objection to fiber bundle substantivalism (Gomes and Butterfield 2023a; Gomes and Butterfield 2023b; Healey 2001; Lyre 1999; Stachel 2014). If gauge-related configurations are interpreted as representing distinct physical realities, the same type of indeterminism that arises in general relativity reappears in gauge-theoretic settings.

Historically, the hole argument was first introduced by Einstein during the development of general relativity, when he initially believed that general covariance threatened determinism. Although Einstein later recognized that the argument rested on a misunderstanding, it subsequently became central to philosophical debates about the ontology of space-time[96]. Modern analyses have refined and clarified the argument in various ways (for a comprehensive review, see (Norton, Pooley, and Read 2023)).

In the space-time case, the challenge for substantivalism arises from the apparent existence of multiple mathematically distinct yet observationally indistinguishable models of the universe. Consider a region of space-time — the "hole" — in which all non-gravitational fields vanish. Given a solution to Einstein's equations, one may apply a diffeomorphism that acts nontrivially only inside this region while leaving the exterior unchanged. Because general relativity is diffeomorphism invariant, the transformed configuration is also a solution of the field equations and agrees with the original solution everywhere outside the hole. However, if space-time points possess an identity independent of the fields defined on them, the two solutions must represent distinct physical possibilities, despite being empirically indistinguishable.

In their influential modern formulation, Earman and Norton (1987) argue that this consequence generates a severe form of indeterminism: specifying all physical data outside the hole, together with the dynamical laws, fails to determine uniquely what occurs within it. The theory would therefore permit multiple physically distinct futures compatible with identical initial conditions. Since such indeterminism appears purely artefactual — arising only from representational redundancy — many philosophers conclude that substantivalism attributes excessive physical significance to mathematical structure and invoke strategies as sophistication (see e.g. Pooley 2006; Pooley 2013; Stachel 2014).

The extension to gauge theories proceeds in close analogy. Gauge transformations that act trivially outside a bounded region but nontrivially within it generate gauge-related field configurations that satisfy the same equations of motion and yield identical observable quantities. A fiber bundle substantivalist, who interprets bundle points or fibers as independently real, would seemingly be committed to treating these configurations as physically distinct worlds. The result is again a form of indeterminism grounded solely in descriptive redundancy. The hole argument thus pressures the substantivalist to explain why gauge-related models should not correspond to different physical realities, or alternatively to restrict ontological commitment to gauge-invariant structure alone.

As will become clear, this dilemma motivates relational or structural interpretations of gauge theories, according to which only equivalence classes under gauge transformations — rather than individual bundle configurations — represent genuine physical states.

### 16.4 Relative or intrinsic geometrical objects?

In order to shed more light on the lessons we can learn from the fiber bundle formalism, let us further unravel its conceptual content. Classical matter fields are vectors of a (complex) vector space $V$ associated to each points of space-time $M$. As such, some of the considerations used to describe vectors (as e.g. velocities) living in the tangent bundle of space-time must identically apply to them.

[96] See also Stachel (2014) for a discussion of the related and crucial concept of "point coincidence argument".

Based on such considerations, Catren (2008) proposes a useful terminology to talk about the geometric features of gauge theories, we shall use this terminology throughout the rest of this work. Furthermore, the author state that **internal relativity** and **background independence** are the two independent foundational principles of gauge theories. Let us summarize this point here.

We stated that one of the conclusions of the geometrical interpretation is that the matter field $\psi$ is the coordinate of a geometrical object $\Psi = [p, \psi] \in \mathcal{E}$ where $p = (x, e) \in P$ is a choice of frame and $\psi \in V$ is a vector. $p$ can be identified with a basis $e_i$ of the vector space $V$ such that we can write $\Psi = \psi^i e_i$. The example to keep in mind is the matter field of electromagnetism $\psi \in \mathbb{C}$ as illustrated in B, in which $e_r$ defines the direction of the real axis of the complex plane (and thus the origin of the phase). However, considering only the associate bundle as it is, it is impossible to distinguish a change of frame $e \to e' = eh$ from a change of $\psi \to \psi' = \rho(h)\psi$. That is, it is impossible to distinguish between a gauge transformation and a physical change of the field $\psi$ expressed in the same basis $e$. This is due to the nature of the equivalence relation $\sim$ defined to build $\mathcal{E}$, which leads to $(ph, \psi) \sim (p, \rho(h)\psi)$ and as such the two objects are equivalent to the unique geometrical object $\Psi = \rho(h)\psi^i e_i$. This problem is well known, and is known in physics as the differentiation between active and passive transformations, or between coordinate transformations and automorphisms in a mathematical jargon. This has important consequences regarding the principle of identity of indiscernible[97] (PII) applied to matter fields as objects living in $\mathcal{E}$. Indeed, the PII fail to work in $V$ as "two vectors are different and indiscernible geometric objects" (p.6) (Catren 2008). Elements of $V$ are thus named **relative geometric objects**[98] as they lack intrinsic properties and they can thus only be defined relatively to other vectors. This is the reason why the use of frames $e_i$ are so necessary in the whole discussion of gauge theories: they allow us to compare relative geometric objects.

In gauge theories, matter fields $\Psi$ are section of $\mathcal{E}$ and thus associate the value of a relative geometric object at each point of a region of space-time $U \subseteq M$. Considering a group element $h \in G$, the relation $[ph, \psi] = [p, \rho(h)\psi]$ tells us that it is impossible to distinguish a frame transformation i.e. a gauge transformation from a transformation of the vector i.e. a bundle automorphism. The property that elements of $\mathcal{E}$ are relative geometric objects is called **internal relativity**. Internal relativity forces us to reconsider our seductive interpretation of gauge transformations as (passive) frame transformations as we have no definite way to rule out an active interpretation of gauge transformations in the geometric framework. An active interpretation of gauge transformations would consider that gauge related fields represent different but physically equivalent physical situations. This debate on the distinction between an active or a passive transformation within a geometrical context joins the already vividly discussed problem of the interpretation of diffeomorphism and general covariance in general relativity (see e.g. Earman 2006; Norton 1993).

While all fibers of $\mathcal{E}$ are isomorphic to the same vector space $V$, there is no canonical way to identify or compare two geometrical objects, say $\Psi_1$ and $\Psi_2$ living in internal spaces $V_{x_1}$ and $V_{x_2}$ associated to two different points of $M$ say $x_1$ and $x_2$. This fact is a direct consequence of $\Psi_i$ being relative geometric object i.e. of internal relativity. This impossibility to identify canonically elements of different fibers is called **fiber bundle locality**.

The introduction of a connection on $P$ and its induced action on $\mathcal{E}$ allows to compare vectors living in different vector spaces by identifying the frames in infinitesimally nearby spaces. However, Catren (2008) argues that the presence of a connection allows for such an identification while preserving fiber bundle locality. Indeed, a connection defines how to drag a frame from one fiber to another and how to derivate vectors but still it does not break the equality $[ph, \psi] = [p, \rho(h)\psi]$ as long as $A$ transform accordingly to Eq. 142. This allows us to interpret vertical bundle automorphisms as active transformations in which both the geometrical object $\Psi$ and the connection $A$ change together in order to leave the notion of horizontality and covariant derivative of vectors invariant. As such, the active interpretation of gauge transformations must be accounted for, and is deeply important in order to find a consistent geometric interpretation of gauge theories. It forbids us to simply deem gauge transformations as invariance under arbitrary change of frames associated to the fiber. For a

[97] See Appendix C.2 as well as our contribution: https://leovacher.github.io/files/Philo_maths_Vacher.pdf

[98] On the other hand, vectors could be promoted to intrinsic geometric objects if each of them were associated to a unique property as a color or a name.

discussion on the importance of active gauge transformations in the attempt to extract philosophical consequences from the geometric formulation of gauge theories, see also Guay 2004.

This idea gives us a new view at the gauge principle which can be understood as the requirement for the theory to preserve fiber bundle locality.

Adding a kinetic term for the curvature of the connection renders the connection itself – and thus the whole geometric structure – dynamical. This requirement is referred to as **background independence**[99]. Background independence is an additional requirement to be asked independently of fiber bundle locality for gauge theories.

As such, the locality requierements makes it impossible to treat matter fields as intrinsic geometrical objects given at each point of space-time. From this conclusion, G. Catren proposed to use the "homotopic paradigm" in order to further explore the nature of the identity of objects laying on the same gauge orbit (Catren 2022; Catren 2023). This approach can be understood as a refined way to perform a sophistication and we shall come back to it in Sec. 17.5.

Most of the above conclusions also apply to velocities, or more generally to vectors in the tangent bundle of general relativity, although they hold even more strongly in the case of gauge vectors. In particular, without a connection it is impossible to identify velocities at different points of space-time, and this identification is path-dependent (fiber bundle locality). Furthermore, the components of a velocity can only be specified relative to other vectors forming a frame (internal relativity). However, velocities can be defined geometrically as tangent vectors to a worldline $u : \gamma(\tau) \to \mathrm{d}\gamma/\mathrm{d}\tau$. This direct relation to curves in space-time makes them more natural candidates for intrinsic geometric objects than matter fields, whose values lie in internal vector spaces not canonically tied to the space-time manifold. This disanalogy ultimately stems from the existence of a solder form (see Sec. 14.1), which provides a canonical relation between the frame bundle and the tangent bundle in general relativity. In this respect, the situation in gauge theory is even stronger: within the purely geometric framework, there is no intrinsic criterion allowing one to distinguish a frame transformation from a transformation of the vector itself. Finally, the familiar claim that "velocities are relative" usually concerns observer-dependence of the measured (3-)velocity—i.e. how different observers compare tangent vectors—not the geometric status of the tangent vector $u$ itself, which remains an intrinsic object along the worldline even though its components vary with the chosen frame (see our discussion in Sec. 19).

### 16.5 Gauge as a handle for couplings

Rovelli's slogan that "gauge is more than mathematical redundancy" can look, at first glance, like a challenge to this traditional anti-substantivalist moral (See Sec. 12.6). His central claim ("Couple") is that gauge-dependent quantities encode information about how a system *can* couple to other systems. The point is not merely pragmatic. Rovelli 2014's parable of two fleets of spaceships is designed to exhibit a precise structural fact: when two gauge systems that are separately describable by gauge-invariant data are brought into interaction, the composite system generically has an additional gauge-invariant observable that cannot be constructed from the gauge-invariant observables of each subsystem alone; it is instead built from gauge-dependent variables of each subsystem. In this sense, the gauge-dependent variables function as "handles" for coupling: they parameterise possible inter-system relations that only become physical when a second system supplies the missing reference. This interpretation build a similarity between gauge dependent quantities and relative objects as velocity, which acquire meaning only when measured with respect to another system.

Let's see how Rovelli 2014 extend this idea to a gauge theory in the fiber bundle framework. In his example, Rovelli takes the example Yang-Mills SU(2) theory, positing a symmetry between neutrons $n$ and protons $p$. Of course, the same analogy should be taken for the phase of the wave-function in electromagnetism, or the color of a particle in chromodynamics. Rovelli's position is stated clearly by himself: "*Suppose we observe one fermion at a point $x$. Can we say if it is a $p$ particle or an $n$*

[99]This terminology echoes with the notion of background independence in general relativity, which is the fact that the metric $g$, defining length and angles at each point of space-time (in each tangent space), is not fixed but the solution of a dynamical equation. The two concepts do not conflate. As discussed in Sec. 14.1, requiring $F$ and thus $A$ to be dynamical still keep the metric $\langle.|.\rangle$ fixed in each internal space, and do not induce dynamics for the space-time metric (entering via the Hodge star in $\mathrm{Tr}(F \wedge \star F)$).

*particle? The answer of course is subtle. A priori, there is nothing that distinguishes the two, so, observing one particle we can arbitrarily call it p or n as we wish. But once one particle has been named, any other particle at a point y connected by a given path $\gamma$ to the first is uniquely determined as a given combination of p and n. This is, of course, because the gauge field defines a parallel transport operator along the path, which allows for the angle between the two particles' states $\psi_x$ and $\psi_y$ in the internal space to be well defined with respect to one another. In other words, the quantity*

$$\theta = \langle \psi_x | \, Pe^{\int_\gamma A} \, | \psi_y \rangle \tag{62}$$

*is gauge invariant. Here we see again how a gauge invariant quantity is a relational quantity, relating gauge field variables and two gauge fermion variables. The nature of a particle, whether it is p or n, is something which is defined in relation to the field and other particles, in the same manner in which a direction in spacetime is only defined with respect to the gravitational field and something else. These relations are cleanly expressed in the geometrical representation of Yang-Mills theory in terms of fiber bundles. In this geometrical picture, the connection is a field that establishes a preferred relation between a frame in a fiber and a frame in a nearby fiber. In other words, it is a quintessential relational observable. Gauge invariance is then the freedom of choosing a frame in each fiber. This geometrical language emphasizes the similarity between Yang-Mills theory and general relativity. In both cases gauge invariance is a matter of arbitrary coordinates. As I emphasize below, in both cases the choice of a particular gauge can be realized physically via a coupling: with a material reference systemic general relativity, with the choice of reference fermions in Yang Mills theory. Therefore the geometry of the bundle describes the possibility of coupling for the gauge field. Gauge theories are sometime introduced mentioning the historical idea of promoting a global symmetry to a local one. The purpose of the field would be to realize the local symmetry. This idea, however, leaves the question I am addressing in this note open: why do we need to describe the world with local symmetries if we then interpret these symmetries as mathematical redundancy?*" (Rovelli 2014 p.4). And later (p.5): "[...]*a detector that recognizes the p particle in the original Yang Mills theory is in principle possible; what it is actually measuring is the angle in internal space between the measured particle and a p standard in the detector itself.*".

Crucially, however, Rovelli's proposal is *not* a licence for naive bundle substantivalism. Rovelli's own measurement story pushes in the opposite direction. A gauge variable, he argues, can be *measured* yet cannot be *predicted* from the subsystem's dynamics alone: its value is fixed only relative to an external reference system that participates in a gauge-invariant interaction with the subsystem. Using the concept of "partial observables", gauge variables are precisely those quantities whose values are accessible at the boundaries of an interaction but remain underdetermined by the isolated system's equations of motion (Rovelli 2002). This means that the physical significance of gauge-dependent quantities is irreducibly relational: the numbers one assigns to them do not describe intrinsic properties of a single entity, but relations between entities (system-apparatus, subsystem-subsystem, etc.). Rovelli's core moral is therefore better read as: *gauge expresses the relational structure required for coupling*, rather than *gauge-dependent values name additional intrinsic furniture*. It is a threat for substantivalism, as if this claim holds, we might wonder why positing the existence of so many entities (parsimony principle) if only their relations are physically significant.

Teh 2013's response both sharpens and constrains this relational moral. Teh agrees that "Couple" captures a genuine insight, but argues that Rovelli's parable does not transfer straightforwardly to General relativity and Yang-Mills-type theories. In realistic local gauge theories, the relevant "subsystems" are not two independent copies of the same kind of mechanical model. In particular, when gauge fields are coupled to matter, the standard geometric picture treats matter fields as sections of *associated* bundles constructed from the same principal bundle. This undermines the parable's key presupposition that one can first define $S_1$ and $S_2$ independently, list their gauge-invariant degrees of freedom, and only then see a "surplus" appear upon coupling. The coupling structure is already built into the shared bundle geometry: the gauge field $A$ mediates the relation between matter degrees of freedom at different spacetime points, and gauge-invariant quantities (e.g. Wilson lines/loops) combine gauge-dependent ingredients from *both* matter and gauge sectors.

Even more importantly for bundle substantivalism, Teh foregrounds a descent/gluing perspective: on a good cover $\{U_\alpha\}$, one may describe gauge theory by local potentials $A_\alpha$ on patches together with transition functions $g_{\alpha\beta}$ on overlaps $U_\alpha \cap U_\beta$, satisfying compatibility conditions. On this view, the principal bundle is not a primitive arena of additional point-objects; it is the global object reconstructed from the network of relations between local descriptions. That shift matters metaphysically. If the global bundle is obtained by gluing local data subject to overlap relations, then the most natural realist reading targets precisely those relational/gluing structures (including global/topological features), not the haecceitistic identity of individual fibre points. Teh's lesson thus aligns with a *structural* reading of bundles: gauge is the apparatus that makes relations well-defined and globally coherent, rather than a sign that there are extra intrinsic properties living at points of $P$.

In a later paper, Rovelli 2020 integrates these remarks and sharpens his 'handles for coupling' analogy by showing that, when a gauge theory is localized to a subregion, the degrees of freedom that encode how it couples to its complement live at the boundary and are generically not gauge-invariant for the subregion alone—connecting the coupling moral directly to edge modes and boundary charges.

Taken together, these papers motivate a precise anti-substantivalist slogan: *gauge is a relation between couples*. There are (at least) three interlocking senses of "couple" here: (i) **system–apparatus coupling** (Rovelli): a gauge variable gains determinate measurable content only in a joint, gauge-invariant interaction where the apparatus supplies a reference value; (ii) **matter–gauge coupling** (Teh): in realistic Yang–Mills theories, matter degrees of freedom are not independently specifiable apart from the bundle/connection structure that relates internal orientations across spacetime, and gauge-invariants typically combine gauge-dependent ingredients from both sectors; (iii) **patch–patch coupling** (Teh's descent picture): global bundle structure is assembled from local data coupled by transition relations on overlaps, making the fundamental representational work explicitly relational and compatibility-based.

These relational readings challenge naive fibre-bundle substantivalism in the specific way the generalized hole argument diagnoses: they undercut the idea that gauge-related assignments of gauge-dependent values correspond to distinct physical possibilities differing in intrinsic facts about fibre points. Rovelli's "handles" are not intrinsic labels on points of $P$; they are parameters of possible relations to other systems. Teh's associated-bundle and descent perspectives further weaken any haecceitistic reading of fibre points by making the relevant structure manifestly dependent on global gluing and on how matter is geometrically tied to the principal bundle.

The natural moral is not that the fibre-bundle formalism is dispensable, but that its most defensible realist reading must be *sophisticated*: one may be realist about the relational/structural content encoded by connections (and by gauge-invariant quantities built from them) while rejecting the naive substantivalist idea that different gauge-related descriptions correspond to different intrinsic distributions of properties over individuated fibre points. In this sense, the Rovelli–Teh debate does not rehabilitate naive bundle substantivalism; it helps explain why gauge structure is ubiquitous *without* treating gauge redundancy as a catalogue of extra intrinsic facts. For further discussion in a quantum setting, see also Rivat 2023.

## 17 Fiber bundle meets metaphysics: properties

As we saw, the fiber bundle formalism, provides a mathematically precise framework in which many physical quantities are represented not as intrinsic properties of spacetime points, but as geometric structures defined over a base manifold. If one take fiber bundle seriously as a fundamental feature of gauge theories, this perspective has been identified as having potentially significant implications for the metaphysics of properties, particularly for views that treat properties as pointwise intrinsic features instantiated independently at each spacetime location.

A useful concept is what Butterfield 2006 calls **pointillisme**: the idea that the fundamental physical facts are realised by (i) intrinsic properties instantiated at space-time points (or point-sized regions), together with (ii) spatiotemporal relations among those points. One reason metaphysicians are drawn to this picture is that it offers a straightforward account of **resemblance**: if two points (or two objects) instantiate the same property, then they resemble in that respect. In the background sit familiar metaphysical devices: **universals** (repeatable entities wholly present in each instance), **tropes** (particularized property-instances), or Lewis' notion of **natural** (sparse) properties, all intended to underwrite objective similarity and difference, as opposed to merely linguistic grouping. For definitions, see Appendix C.2.

A closely related, but stronger, thesis is Lewis' **Humean Supervenience** (HS) (Lewis 1986). Whereas pointillisme is mainly a claim about *what the basic physical facts are* (point-by-point intrinsic properties plus spatiotemporal relations), HS adds a claim about *what fixes everything else*: once the complete "Humean mosaic" is fixed—i.e. the full distribution of local qualitative matters of fact throughout space-time, together with the spatiotemporal arrangement of those facts—then *all* further facts supervene on it. In particular, HS aims to make laws of nature, causal relations, chances, and counterfactual truths depend entirely on the mosaic, rather than requiring any additional irreducible governing structure. The motivation is to avoid "extra" necessities in nature: if two possible worlds agree point-for-point in their local qualities and their spatiotemporal relations, then, according to HS, they must agree on everything, including which laws obtain.

Gauge theory, interpreted geometrically, makes this attractive picture harder to sustain. The reason is not (only) quantum mechanics or non-locality, but the way gauge theories force us to represent fields as sections of bundles equipped with connections, rather than as globally comparable assignments of values. In standard textbook presentations, a (classical) field is introduced as a map from space-time into a single space of values $\phi : M \to V$, where $M$ is the base manifold (space-time) and $V$ is a global *value space*. This representation invites a pointilliste reading: the field "has a value" $\phi(x)$ at each point $x$, and the same value can, in principle, occur at distinct points.

The fiber bundle formulation replaces this global value space by a *localised* one. Instead of a single $V$, we have a bundle $\pi : E \to M$ whose fiber $F_x = \pi^{-1}(x)$ encodes the possible local states at $x$. A field is represented by a *section* $\sigma : M \to E$ with $\pi \circ \sigma = \mathrm{Id}_M$. On this picture, $\sigma(x) \in F_x$ and $\sigma(y) \in F_y$ live in *different* fibers. Unless we add further structure, there is no canonical sense in which $\sigma(x)$ and $\sigma(y)$ can be said to be "the same value", because there is no canonical identification $F_x \simeq F_y$.

The additional structure that mediates comparison is a *connection*. In geometric terms, a connection specifies which directions in the total space $E$ count as "horizontal" (i.e. as preserving internal state) and thereby defines *parallel transport*: given a path $\gamma$ in $M$ from $x$ to $y$, the connection determines a map (typically path-dependent) $\Pi_\gamma : F_x \to F_y$ that tells us how to transport a local state at $x$ to a comparable local state at $y$. The key point for metaphysics is that the connection does not merely encode dynamics: it is what makes cross-point comparisons of internal states *available at all*. This is the bridge to the metaphysical debate about properties. Such a discussion is key to find the proper way to understand what it could mean and imply to be realist about fiber bundles.

## 17.1 Maudlin: gauge structure as an alternative to resemblance-based metaphysics

One of the first authors to propose a revision of the key metaphysical concepts through the fiber bundle formalism, is Tim Maudlin 2007. His starting point is straightforward: if metaphysics is supposed to tell us what exists and how reality is structured, then it should take guidance from our best physics rather than from the surface grammar of ordinary language. He thinks philosophers often make a mistake here: they treat features of our descriptions (for example, the subject-predicate form of sentences, or the shape of a mathematical formalism) as if they directly revealed the structure of the world.

This motivates Maudlin's criticism of a popular Quinean strategy. Quine proposes that we can discover what a theory is committed to by translating its claims into a precise logical language and then looking at what the theory must quantify over (what sorts of things must exist for the theory to come out true) (Quine 1953). Maudlin's reply is that the crucial step is exactly the one Quine treats as routine: translation. The very same ordinary sentence can be translated in different ways, and these different translations can require different things to exist. So the "read off the quantifiers" test cannot, by itself, settle deep questions about what the world contains; we also need to say what in the world makes our predicates true (what it takes for "is a sibling", "is red", etc. to be true).

This matters because many metaphysicians introduce *privileged* properties to explain why similarity is not arbitrary and why laws do not treat all predicates equally. For Lewis 1986, this takes the form of *natural* (sparse) properties within a broadly picture of Humean supervenience. For Armstrong 1978, it takes the form of a realist theory of *universals* tied to scientific explanation. Without some such privileged structure, one risks a picture in which the world contains objects but no objective pattern beyond their mere number (cardinality). Maudlin then targets an assumption that often comes with these views: that there are *metaphysically pure* properties and relations. A relation would be metaphysically pure if it could hold between two things in a world where only those two things exist; a property would be metaphysically pure if a lone object could have it all by itself. Maudlin argues that physics gives us strong reasons to doubt the existence of such purity.

Maudlin 2007's first example is distance. Distance between two points is not a simple relation that can exist "in isolation", because it is defined in terms of lengths of paths connecting the points (distance is the minimal path length). If there were no connected space providing such paths, distance would not even be well-defined. His second example is the comparison of directions. On curved spaces, directions at different points live in different tangent spaces, and the only way to compare them is by specifying a connection and transporting one direction to the other along a path. Different paths can yield different results, so there is no path-independent fact of "pointing the same way". Both notions cannot be metaphysically pure.

Maudlin's claim is that gauge theories generalize this situation. In the fibre bundle formulation, each space-time point has its own internal space of possible states (a fibre), and there is no built-in notion of "the same internal state at two different points". A connection supplies a rule for comparing states by parallel transport along paths. Therefore, many quantities that might be naively treated as pointwise intrinsic properties (Maudlin's example is non-Abelian colour charge) do not support an absolute, path-independent notion of sameness across space-time. This is why Maudlin thinks gauge theory pressures traditional resemblance-based metaphysics: the fundamental structure looks less like objects bearing shareable point-properties, and more like global geometric structure encoded by bundles and connections.

### 17.2 Muntean: from formalism to ontology requires further commitments

A detailed response to Maudlin is provided by Muntean 2011. The author finds Maudlin's project interesting and largely sympathetic in spirit: he agrees that metaphysics should take modern physics seriously, and he agrees that Maudlin's critique is not aimed only at Lewis' Humean Supervenience. Instead, Muntean stresses that Maudlin's argument is meant to challenge any metaphysical view that tries to explain similarity between things by saying that they literally share the same property (for example, by sharing a universal, or by resembling in virtue of tropes).

However, Muntean argues that Maudlin's positive step goes too fast. Maudlin does not only claim that gauge theory makes some familiar metaphysical ideas harder to maintain; he also suggests a new picture in which we should replace the standard talk of shared properties and resemblance with the geometric framework used in gauge theory. Muntean calls this proposed replacement "fiber bundle metaphysics" and argues that it is promising but not yet strong enough to reform metaphysics.

Muntean's first worry is methodological: the fiber-bundle framework is, in the first instance, a mathematical way of representing gauge theories. Mathematical tools can be extremely useful in physics without automatically describing what the world is fundamentally made of. Because of this, Muntean thinks Maudlin owes the reader a clearer reason to treat the fiber-bundle framework as more than a convenient representation. In other words, if one wants to build metaphysics directly from the fiber-bundle formalism, one must explain why that particular formalism should be taken as revealing the world's basic ontology, rather than being a powerful "map" that helps us do calculations and organize the theory.

A second set of worries concerns what Muntean calls the metaphysical "loneliness" of objects. In much traditional metaphysics, it is meaningful to ask whether an object could exist all by itself and still have certain intrinsic features. Maudlin's argument against "pure" relations and properties suggests that many features we care about only make sense in a world that already contains a connected space-time with additional geometric structure. Muntean points out that this can look like a change of the rules rather than a direct refutation: if one insists that the background space-time and its structure must always be present, then many standard metaphysical scenarios (like lonely objects in possible worlds) are excluded from the start. Muntean suggests that Maudlin should say more about how much of this is a discovery forced on us by physics, and how much is a choice about what counts as a genuine metaphysical possibility.

Closely related is a worry about identity and counting. Gauge theories often allow many different mathematical descriptions of what seems to be the same physical situation (this is the usual "gauge freedom"). Muntean argues that if Maudlin wants to treat the fiber-bundle structures as the basic ontology, he needs to be clearer about how we should identify and count those structures: which mathematical differences represent real physical differences, and which are merely different representations of the same situation.

Muntean also offers physics-motivated replies to Maudlin's central claim that comparisons between distant states are necessarily path-dependent, and that therefore there can be no fundamental relation of "being the same" across space-time. He suggests that one can often recover meaningful local comparisons by focusing on quantities that do not change under the relevant transformations (for example, certain invariant features of the geometric structure), and he notes that in many geometrical settings one can use specially preferred curves (geodesics) to define comparison procedures. The point is not that Maudlin's observations about path-dependence are wrong, but that they do not by themselves force the radical metaphysical conclusion that resemblance and intrinsic structure must be eliminated. Rather, they may show that any notion of similarity in gauge settings involves an element of convention and depends on background geometric structure.

Finally, Muntean emphasizes a limitation of the fiber-bundle program as a general metaphysical replacement. The same mathematical language of fiber bundles can be used in different ways in different physical theories. In particular, the sort of bundle structure used for internal gauge symmetries is not straightforwardly the same as what is at stake in general relativity, where the key symmetries concern transformations of space-time itself. Muntean argues that this "double life" of fiber bundles complicates any claim that one single fiber-bundle ontology automatically unifies the metaphysics of all fundamental physics. If Maudlin's goal is a general reform of metaphysics, then this mismatch must be addressed more carefully.

Muntean's overall verdict is therefore cautious: Maudlin's proposal is stimulating and may be on the right track in urging metaphysicians to learn from gauge theory, but it does not yet provide the interpretive and ontological argument needed to replace traditional metaphysical accounts of properties and resemblance. In Muntean's words, Maudlin's project is promising, but "not powerful enough to reform metaphysics".

### 17.3 Arntzenius: realism about bundle structure without eliminativism about properties.

Arntzenius 2012 accepts a central geometric lesson that Maudlin also emphasizes. In modern geometric theories, one cannot in general assume that there is a primitive, location-independent fact of the matter as to whether a quantity instantiated at one point is literally *the same* as a quantity instantiated at another. Arntzenius introduces this point by recalling the case of tangent vectors in curved spacetime: since tangent vectors at distinct points live in distinct tangent spaces, their comparison requires additional geometric structure, namely a connection, and is in general path-dependent. Gauge theories, on his view, generalize exactly this phenomenon. At each spacetime point there is a fibre whose points represent the possible states of the relevant field at that point, and a field configuration is represented by a section selecting one point in each fibre. Hence, the naive textbook picture in which one and the same field-value is simply repeated at different spacetime points is replaced by a picture in which values are localized to distinct fibres and can be compared across spacetime only by means of the bundle geometry.

However, Arntzenius does not draw from this the stronger conclusion that gauge theory undermines ordinary property-talk altogether. On the contrary, he explicitly resists Maudlin's more revisionary moral. His claim is that gauge theory forces us to revise which features count as the fundamental shared properties and relations, not to abandon the idea of shared properties as such. The basic mistake is to identify the fundamental properties with naive local field-values. Once the theory is formulated in fibre-bundle terms, those are no longer the right candidates for repeatable universals. But this does not mean that there are no universals left. Rather, the fundamental universals are, according to Arntzenius, occupation properties and geometrical relations of the fibre bundle. Different locations can genuinely have in common such features as being occupied or unoccupied, and can stand in the same geometrical relations, even if there is no primitive, path-independent fact of sameness for the fine-grained field-values themselves.

This is why Arntzenius's positive proposal is best understood as a form of fibre-bundle substantivalism. Gauge theories are, on this view, theories about the geometrical and occupation structure of fibre-bundle spaces. The fibre bundle is not merely a representational convenience, nor is it something to be eliminated in favour of a reduced ontology of gauge-invariant quantities alone. Its points and its local geometrical structure are taken seriously. Sections represent field states, and connections represent further geometrical structure of the bundle. In this sense, Arntzenius is a realist about the full bundle formalism.

This position is identified by Jacobs 2023 to the position known as **locationism** defined later in the metaphysics literature by Wolff 2020. The basic idea is that the fine-grained content of a gauge field is to be understood by analogy with location: what matters fundamentally is which points of the relevant fibre-bundle space are occupied, together with the geometrical relations obtaining among those points. So the ontology is not one of primitively repeatable field-values instantiated at many spacetime points; rather, it is one of occupation facts in a structured bundle space.

Arntzenius then discusses how to escape the problems inherent to gauge redundancy in this framework. Gauge theories admit many mathematically distinct but observationally equivalent descriptions, and one might therefore think that any realist interpretation of the bundle structure is untenable: if gauge-related bundle states represented distinct physical possibilities, then one could generate hole-style cases of apparent indeterminism. Arntzenius's response is that this anti-realist conclusion does not follow. First, one could simply be a sophisticated fibre-bundle substantivalist and deny that gauge-transformation-related bundle states represent genuinely distinct possible worlds. Second, even if one did regard them as distinct possibilities, one need not thereby infer a genuine failure of determinism. In other words, the mere existence of gauge-related alternatives does not yet show that the laws leave the future physically open: it may only show that the same evolution can be described in more than one formally distinct way. And third, he is generally sceptical of arguments that move too quickly from intuitions about symmetry-related possibilities to conclusions about ontology. For him, questions about ontology should be settled primarily by the simplest and most natural overall theory, not by quick appeals to symmetry-based intuitions about possible worlds.

### 17.4 Jacobs: sophistication plus anti-quidditism and a realist recovery of gauge properties.

Jacobs 2023 develops a *sophisticated* response to gauge symmetry: gauge-related models should not be interpreted as representing distinct physical possibilities, but as different mathematical representations of one and the same physical situation. In this respect Jacobs' proposal is closer to Arntzenius and Weatherall than to reductionist accounts, since his aim is to dissolve rather than merely tolerate the apparent underdetermination and indeterminism generated by gauge symmetry. But this strategic similarity should not obscure a deeper disagreement. Jacobs explicitly rejects the *locationist* interpretation associated with Arntzenius. On that picture, the fine-grained content of a field theory is understood in terms of the occupation of points in a fibre-bundle space, so that field-values are treated as analogous to locations in a value space. Jacobs argues that this picture breaks down in the case of field theories. Since a field is represented by a section assigning a value to each spacetime point, and since spacetime points are themselves already locations, it is obscure to say that spacetime points are additionally "located" in a further field-value space. He therefore replaces locationism with what he calls **associated bundle Platonism**: the elements of the associated bundle (i.e. pairs of frames/coordinates associated to an equivalence class) represent physically real field-values, and sections of that bundle represent genuine physical field configurations $\psi$ for matter. As such, this reading favors the associated bundle as the relevant mathematical object for reification.

In the textbook picture, it is natural to think of a field as assigning to each spacetime point a value drawn from one common value-space. Jacobs stresses that the fibre-bundle picture replaces this with a more local conception. Each spacetime point carries its own fibre, that is, its own local space of possible field-values. As a result, there is no automatic, path-independent answer to the question whether the value instantiated at one point is literally the same as the value instantiated at another. Comparisons across points require additional geometric structure, in particular a connection, which induces relations of parallel transport. The moral is not that field-values disappear, but that they are *localised*: they are attached to fibres, and the relations between them are encoded in the geometry of the bundle. This is the background to Jacobs' disagreement with Maudlin. Maudlin takes the locality of fibres to undermine the category of properties: if distinct spacetime points cannot literally instantiate the same field-value, then, he argues, there is no genuine notion of similarity left for gauge quantities. Jacobs rejects this conclusion. His complaint is that Maudlin treats strict identity as the only possible basis of similarity. But in ordinary cases this is too demanding. A 1 kg object and a 2 kg object do not possess the same determinate mass, yet they are more similar to one another with respect to mass than either is to a 100 kg object. The reason is that mass-space has structure. Jacobs argues that the same is true in the gauge case. Even if field-values at distinct points are numerically distinct, the associated bundle still contains rich structural relations between them, such as parallel transport and group-theoretic relations within fibres. So the fibre- bundle formalism does not force us to eliminate gauge properties; rather, it invites a different metaphysics of them.

The first step of Jacobs' positive view is therefore ontological. According to associated bundle Platonism the associated bundle is not merely a mathematical device. Its elements represent physically real values of the matter field, and a section of the associated bundle represents a physical assignment of such values over spacetime. Spacetime points instantiate field-values somewhat as particles instantiate masses. However, because each spacetime point has its own fibre, values instantiated at distinct points are numerically distinct. Strict cross-point identity of field-values is therefore not available. What the bundle supplies instead is a network of higher-order relations between localised values. In particular, it gives us physically meaningful facts about how values at different points are related by parallel transport along paths.

The second step is modal rather than ontological. Jacobs argues that sophistication can succeed only if one adds **anti-quidditism** about gauge quantities. In his usage, anti-quidditism is the claim that determinate field-values do not possess primitive identities independently of the total pattern of structural relations in which they stand. Field-values are not bare thisnesses; they are qualitatively individuated by their place in a web of relations. This point is crucial because a gauge transformation generally does change which element of the fibre a section assigns to a given spacetime point. If those bundle-elements had primitive identities across models, gauge-related sections would have to represent different physical fields. Jacobs denies that consequence. Since gauge transformations preserve the relevant vertical and horizontal structure of the bundle, anti-quidditism allows us to say that they preserve the physical field-values represented, even though they change the mathematical representatives. This is the sense in which gauge-related models are different descriptions of the same physical situation.

For Jacobs, Yang–Mills fields $A$ cannot be treated in exactly the same way as matter fields, because matter fields are represented by sections of an associated bundle, whereas the Yang–Mills field is represented by a connection on a principal bundle; so, unlike the matter case, one cannot straightforwardly read it as an assignment of field-values to spacetime points. He therefore considers two options. The deflationary option says that the Yang–Mills field is nothing over and above the induced connections on associated bundles, so that what is physically real are local matter-field values plus the infinitesimal relations between them; Jacobs finds this unsatisfactory because it makes it mysterious how distinct matter fields can be coupled to one and the same Yang–Mills field. He therefore prefers an inflationary option: instead of reifying the principal bundle itself, he reifies the bundle of connections, where a Yang–Mills field is represented by a section assigning, to each spacetime tangent direction, the corresponding horizontal structure of the principal bundle. This lets him treat the Yang–Mills field as a genuine physical entity while still preserving sophistication: gauge-related models differ only representationally, not physically. He also says that this whole picture extends straightforwardly to non-Abelian theories, because nothing essential in the argument depends on the Abelian character of U(1).

On this view, then, Jacobs agrees with Arntzenius only at the level of general interpretive strategy. Both reject reduction and both use the fibre-bundle formalism to block the inference from gauge symmetry to underdetermination. But they disagree about the correct metaphysics of field-values. Arntzenius interprets the theory in terms of occupation facts in bundle space; Jacobs instead reifies the associated bundle and treats its elements as genuine field-values. The disagreement is therefore not over whether the fibre-bundle formalism should be taken seriously, but over how it should be taken seriously.

Jacobs finally argues that this metaphysical picture yields a **local and separable** account of the Aharonov–Bohm effect. The matter field evolves locally by following the horizontal structure determined by the connection. In the two-slit experiment, the two branches of the matter field are transported differently along the two sides of the solenoid, so that when they recombine at the screen they occupy different points on the fibre above the same spacetime point. This difference in field-value is what underwrites the shift in the interference pattern. In that sense, the explanation is local: only the connection where the matter field is located determines how the field evolves. It is also separable in Jacobs' sense, since the structure over larger regions is fixed by the local bundle structure over overlapping subregions.

At the same time, Jacobs insists that the Aharonov–Bohm effect remains **holistic** in a limited but important sense. Because value spaces are localised to individual fibres, there is no gauge-invariant answer to the question where, exactly, the phase difference is built up along an open path. One can compare the two branches only when they meet again at the same spacetime point, where they lie in the same fibre. The total phase difference is physically real and measurable, but there is no objective decomposition of this effect into a sum of determinate local phase increments along the way. For Jacobs, this holism is not a defect of the fibre-bundle account, but a direct consequence of the fact that the theory localises value spaces rather than treating them as globally shared.

The broader moral is that symmetries are not mere redundancies to be eliminated. In the fibre-bundle formalism they reveal the correct structure of the relevant physical quantities. This is why Jacobs prefers sophistication to reduction. A reductionist interpretation attempts to replace the original theory with one formulated purely in terms of invariant quantities, thereby discarding much of the bundle structure. Jacobs' sophisticated interpretation, by contrast, preserves that geometric structure while explaining why gauge-related models do not correspond to different physical worlds.

### 17.5 What is identity? Going beyond sophistication with categories

The two approaches of Jacobs uses sophistication to suppress gauge ambiguity, while Arntzenius considers sophistication as a relevant strategy to address gauge redundancy. As we saw, sophistication identifies different situations as identical if related by isomorphisms. It allows thus to identify as identical physical situations different space-time related by diffeomorphisms and different setting related by gauge transformations (vertical bundle automorphisms). This allows to counter some of the arguments againt substantivalism, as hole-type considerations.

Proposition of more advanced positions of what should, or not be identified as physically identical situations have been proposed in the context of category theory. Category theory is especially relevant for the study of gauge theories, as it focuses on defining relevant notions of identities (which can be stronger or weaker) and the study of relationships between classes of objects. The most general definition of isomorphisms invoked in sophistication takes its definition in category theory (See Appendix C.1). Weatherall and Catren both appeal to category-theoretic ideas, but they do so in importantly different ways. For Weatherall 2015, the central point is methodological: questions about gauge, excess structure, and physical equivalence should be recast in terms of categories of models and the arrows between them. On this approach, one does not simply stipulate that gauge-related models are physically identical; rather, one asks whether the relevant functor[100] between formulations is full, faithful, and essentially surjective, i.e. whether it "forgets nothing." In electromagnetism, for example, this means that once gauge transformations are added as morphisms between vector-potential models, the resulting category is equivalent to the category of field strength models, so the alleged surplus is diagnosed in terms of the original choice of arrows rather than as a deep metaphysical problem. Catren 2023 accepts the relevance of categorical and groupoid-theoretic tools, but pushes the lesson

[100] A functor is a transformation between two categories.

much further. In his "homotopic paradigm" reading of Yang–Mills theory, gauge-related configurations are not merely redundant descriptions of a single underlying reality; they are numerically distinct but indiscernible items whose structured family of identifications is itself part of the content of the theory. What matters, then, is not only the objects described by the theory, but also the different ways in which those objects can count as equivalent. In homotopic terms, one begins with points, then considers paths connecting one point to another, then continuous deformations turning one path into another, then deformations between those deformations, and so on. Rather than collapsing all these relations into a single quotient that treats them as mere redundancy, Catren argues that we should keep track of this layered structure of identifications. The resulting picture is a network of reversible equivalences between points, paths, and deformations of paths. Mathematically, such a structure is described by a groupoid — a category in which every morphism is invertible — and, in the richer homotopic setting, by a higher groupoid. Applied to gauge theory, this framework captures the structured equivalences between spacetime points, fibers, and gauge-related configurations. On this view, gauge symmetry is not simply a bookkeeping device for removing representational excess; it expresses part of the intrinsic geometric organization of gauge theories themselves.

### 17.6 Overview

Taken together, these positions clarify what is at stake when one "takes the bundle seriously" in the metaphysics of properties. The fiber bundle framework undermines a naive pointilliste picture in which field values are globally comparable intrinsic properties at points. The remaining open question is how strong the metaphysical moral should be. Maudlin reads gauge geometry as motivating a replacement of resemblance-based property metaphysics by bundle/connection ontology (Maudlin 2007). Muntean argues that this step requires stronger realist and interpretive premises, and that physics leaves room for less radical reconstructions. Arntzenius 2012 and Jacobs 2023 both consider versions of sophisticated realism: gauge redundancy is handled by identifying physical states with equivalence classes of models, while preserving the explanatory centrality (and ontological seriousness) of bundle and connection structure. Arntzenius 2012 defends a position close to fiber bundle substantivalism, in which properties are understood as "locations" and relations on the fiber bundle, just as one can think of positions and relations in space-time. On the other hand, Jacobs 2023 proposes a form of associated bundle and connection bundle platonism, with an anti-quiddistic reading. Beyond sophistication formulations have been proposed using category theory, as a way to identify identical physical situations while preserving the relevant geometrical structures, beyond simply quotienting all gauge equivalent situations in an equivalence class.

## 18 Alternatives to substantivalism: relationalistic interpretations

The previous sections put significant pressure on a naive *fiber bundle substantivalism*. The parsimony concern (Sec. 16.2) suggests that we should not read the entire bundle superstructure as additional unobservable "stuff"; the generalized hole argument (Sec. 16.3) suggests that treating gauge-related models as distinct physical possibilities yields an unacceptable underdetermination/indeterminism; and Catren's diagnosis of *internal relativity* and *fiber bundle locality* suggests that many of the relata (e.g. "field-values-at-a-point-as-shared") are not even well-posed without further relational structure (Sec. 16.4). At the same time, purely eliminativist responses appear to sacrifice part of what makes gauge theory explanatory: its local dynamics, its coupling to matter, and its natural encoding of parallel transport and curvature in terms of a connection. An approach comparable to Rovelli's (Sec. 16.5) suggests that gauge can be understood relationally *precisely because* gauge-variant variables parameterize how subsystems (or subregions) can couple, and because the "missing" information when one isolates a subsystem is typically relational/interface data rather than intrinsic local "stuff".

In this section we defend a relationalist alternative that keeps the fiber-bundle formalism *as the right representational framework* while rejecting the substantivalist reading of its points and fibres. The guiding thought is Jacobs (2023)'s "sophistication plus anti-quidditism" strategy: what matters physically is not the primitive identity of the elements of the fibres (nor the primitive identity of gauge potentials as pointwise objects), but rather the *pattern of relations* encoded by the connection and its induced transport structure, understood *up to gauge isomorphism.* We argue that this yields a natural form of structural realism (ontic or epistemic) about gauge theory—one that also supports a broader sense of "geometry" as *the theory of principled comparisons across local state spaces.* A similar position using the dressing field framework can also be found in Berghofer, François, and Ravera 2025.

### 18.1 Bundle structural realism: the transport structure as the physical core

The pressure from *fiber bundle locality* is that there is no canonical identification between fibres over distinct base points, hence no non-question-begging notion of "the same internal value at different locations". This is exactly what undermines the pointilliste picture of shared local field values and motivates the shift toward bundle/connection ontology (Sec. 17.1). Jacobs' response is to accept the locality moral but resist the eliminativist conclusion about properties by adding *anti-quidditism*: the identity of fine-grained internal values is not primitive; what matters is their role in the relational structure provided by transport (Sec. 16.4). Catren's "relative geometric objects" make the same metaphysical point in a different idiom: fibre elements lack intrinsic thisness and are only individuated relationally. A similar conclusion is also reached following Rovelli's coupling-based path (Sec. 16.5), especially once one takes seriously that isolating a subsystem typically discards relational/interface information rather than intrinsic local facts.

We can now state a thesis that packages these morals into a single interpretative claim:

> **(Bundle Structural Realism / Transport-Structure Realism).** Gauge theories describe objective *relations of principled comparison and transport* between local value spaces, encoded by a principal bundle with connection (or, equivalently, by the induced covariant derivative on associated bundles), where physical states are given by the resulting transport structure *up to gauge isomorphism.* The fundamental ontology is therefore structural: it consists in the network of transport relations (and their curvature/holonomy content, including global/topological constraints), not in substantival fibres or primitive identities of fibre elements.

A clarification is in order about the roles of principal and vector bundles. For many dynamical and coupling questions, it is convenient to focus on a vector bundle $E$ and its covariant derivative $\nabla$. Nevertheless, the principal-bundle level is typically where global features (topological sectors, global form of the gauge group, and the uniform treatment of all associated matter representations) are most naturally represented. The present view therefore does *not* require that principal bundles are eliminable; rather, it is neutral about whether one takes the principal bundle as fundamental or as representationally derivative, provided the realist target includes the full gauge-isomorphism class of the underlying transport structure.[101]

This view is *relationalist* in two connected senses.

**(a) Relationalism about internal degrees of freedom.** Because of internal relativity and bundle locality, the relata in the fibres are not intrinsically individuated; what is physically significant are the relations induced by the connection (how states are compared under parallel transport, how covariant derivatives compare nearby fibres, and how curvature obstructs path-independence).

**(b) Relationalism about gauge redundancy.** The generalized hole argument shows that if we reify all the gauge-related theoretical content, we obtain spurious indeterminism. The sophisticated response is to treat gauge transformations as isomorphisms of models, so that two gauge-related models represent the same physical situation when they are related by an appropriate bundle automorphism intertwining all relevant structure (in particular, preserving $\nabla$ and hence horizontality/parallel transport). That is, when two formulations are related by an isomorphism that preserves the theoretical

[101] Compare the more revisionary claim that one can (in suitable contexts) formulate gauge theory without treating the principal bundle as fundamental, while still recovering a covariant derivative on a vector bundle and addressing "coordination" problems; see Gomes 2024b.

structure, they should not be interpreted as making distinct claims about the world. A further refinement can be done, following Catren (2022) and Catren (2023)'s "homotopic paradigm". The slogan "physical states are gauge-equivalence classes" can suggest a mere set-quotient (an orbit space), but this can erase physically and mathematically significant *identity structure*: distinct gauge transformations can relate the same representatives in different ways, and configurations may have nontrivial stabilizers (automorphisms). On this view, "quotienting by gauge" is best understood as passing to the *gauge groupoid*[102] (or moduli stack) of configurations: gauge symmetries are not extra intrinsic furniture, but the theory's principled account of when and how two descriptions count as the same physical situation. Such an approach may be better suited than sophistication to preserving the global structures that emerge from local transport, as discussed in Sec. 20.1.

The same physical content can be presented in a more explicitly structural way by focusing on transport itself. A connection determines *parallel transport* along paths and *holonomy* along loops. A compact way to express this is to say that a connection defines a *transport functor* from the *path groupoid* of spacetime into the category of internal frames or representations (Schreiber and Waldorf 2007).[103]More concretely: for a principal $G$-bundle with connection, parallel transport along $\gamma : x \to y$ yields an element of $G$ *relative to a choice of frames* (or equivalently an isomorphism between the $G$-torsors over $x$ and $y$).[104] Gauge transformations then act by changing representatives while preserving the underlying transport structure; in category-theoretic terms, they correspond to *natural isomorphisms* between transport functors.[105] This makes the structural realist moral vivid: the basic representational work is done by *comparisons* (arrows) rather than by primitive pointwise internal labels.

This transport-centred reading also clarifies the balance between locality and "holism" in phenomena like the Aharonov–Bohm effect. The theory remains local in the sense that the dynamical evolution of the matter field is governed by local coupling to the connection (via $\nabla$), yet the empirically salient shift is encoded in a global comparison (a holonomy) that is well-defined only for closed paths and not uniquely decomposable into objective local "phase contributions" along open curves. The point is not that the physics is non-local, but that the relevant structure is inherently comparative: it tells us how to relate states at different locations, not what absolute internal values they "intrinsically" possess.

Finally, this perspective provides a direct route to Rovelli's coupling moral in genuinely field-theoretic settings: when a gauge system is split into subsystems (or spacetime regions), the gauge-invariant data of each part typically fails to determine the gauge-invariant data of the whole, and the missing information is relational/interface transport data. In this sense, the transport structure does double duty: it is both the geometric mechanism of comparison and the resource that makes coupling/gluing of subsystems possible without reifying fibre-point haecceities.

### 18.2 Geometry in the broad sense: local state spaces plus rules of comparison

One of the motivations for bundle substantivalism was that bundles *look* like geometry: connections, curvature, parallel transport. Yet the "fiber bundles everywhere" lesson cuts both ways (Sec. 16.1). As Batterman (2003) emphasizes, the same geometric patterns (anholonomy, holonomy, connections) arise in a wide range of physical systems. The Ilinski, Kalinin, and Stepanenko (1997)-style financial analogy makes the point vividly: one can model currencies as local units (gauge choice), exchange

[102]A *groupoid* is a category in which every arrow is invertible. Here the relevant groupoid has as objects the models $(P, A, \Psi)$ (or $(A, \Psi)$ in a fixed-bundle presentation) and as arrows the gauge isomorphisms between them. The corresponding *homotopy quotient* (or moduli groupoid/stack) retains not only the orbits but also the structured ways in which representatives are identified.

[103]The *path groupoid* $\Pi_1(M)$ is the category whose objects are points of $M$ and whose morphisms are (equivalence classes of) directed paths with composition given by concatenation. A *functor* is a structure-preserving map between categories: it sends objects to objects and arrows to arrows, respecting identity arrows and composition. In gauge theory, transport along paths provides precisely such an "arrow assignment".

[104]A *$G$-torsor* is a set equipped with a free and transitive action of $G$ (a "group without a distinguished identity element"), which is often the right way to think about a fibre of a principal $G$-bundle when one does not wish to privilege a particular gauge/frame. See also Appendix D.2.

[105]A *natural isomorphism* between functors is a coherent family of isomorphisms relating their values on each object, compatible with how the functors act on each arrow. Here the coherence condition mirrors the fact that gauge transformations must intertwine transport along every path, not just at isolated points.

rates as a connection, and arbitrage as curvature. The moral should not be that there literally exists a further substantival “money space”, but that the bundle formalism captures a general structural feature: *systems with local conventions and non-trivial comparison rules* naturally exhibit geometric invariants.

This suggests a broader conception of geometry than “the metric structure of spacetime.” In the present context, geometry is the study of structures that (i) assign local state over a base space and (ii) provide principled comparison/transport between them, yielding invariant curvature/holonomy content. While the base space in (i) is usually a space which is geometric, or at least topologic in the traditional sense (a smooth manifold). It usually represents space or space-time and its points are associated to a notion of "location". On the other hand, the space/fiber in which the state lives may be much more abstract. Thus, although this conception of geometry is broader than the usual one, it remains geometric insofar as it is anchored in a geometric base space, typically representing spacetime. The fibers at each point may also be associated with notions such as orientation or length in an abstract internal space, where these notions are defined independently of their spacetime counterparts. Gauge geometry is therefore not a metaphor, but neither is it a commitment to new substances: it is a commitment to a robust *comparative structure* that organizes how local descriptions hang together. This is exactly the sense in which gauge theories are geometrical while remaining compatible with relationalism: the physical world contains patterns of permissible comparisons, not necessarily extra arenas over and above spacetime in which internal entities reside.

## 18.3 Objections and replies

**Objection 1 (“Is this just reduction in disguise?”).** If the physical content is an equivalence class under gauge, why not simply reduce to invariants and be done with it? Reply: sophistication is not eliminativism. The point is to keep the local connection-based structure because it is what formulates the local equations of motion and couplings in the natural way, while still denying that changes of representative correspond to distinct physical possibilities. Reduction may be possible in principle in special cases, but the sophisticated stance explains why the unreduced formalism remains indispensable to practice and explanation.

**Objection 2 (“Structure without relata is empty.”).** A structural realist view may seem to float free of concrete ontology. Reply: the present view need not deny that there are local states and fields; it denies that their identities are primitive *across* fibres. Catren’s and Jacobs’ morals are precisely that the relevant individuality is fixed by relational role within the transport structure, not by intrinsic haecceities. This is a non-eliminativist structuralism: it keeps local physical degrees of freedom while rejecting quidditistic thisness for gauge quantities.

**Objection 3 (“If everything is geometry, nothing is.”).** If the bundle formalism applies to cats and currencies, does “geometric” lose content? Reply: the proposal does not say that any mathematical rewriting counts as geometry. It identifies a specific criterion: the presence of non-trivial transport/comparison structure with curvature/holonomy invariants that constrain and explain the system’s behaviour. That criterion is substantive and is exactly what gauge theories exhibit.

We thus arrive at a relationalist interpretation that preserves the core explanatory achievements of the fiber-bundle formulation while avoiding both ontological inflation and generalized-hole indeterminism. Gauge theories describe a geometry in the broad sense: a transport structure that organizes local state spaces into a coherent whole. The next sections will discuss how such an interpretation can be applied to a simple financial toy model and whether it can be applied to the tangent bundle of generality.

## 18.4 Illustration with Schwichtenberg’s financial toy model

Schwichtenberg’s financial toy model is a useful testing ground for the present transport-structure realism (Schwichtenberg 2019a). The model is simple: countries occupy the sites of a lattice, each country uses its own local currency, and exchange rates relate neighbouring sites. A local change of currency unit acts as a gauge transformation, and arbitrage around a closed loop plays the role of curvature. This makes the example philosophically valuable: it displays in an elementary setting the same formal pattern that, in ordinary gauge theory, is expressed by local frames, connections, and curvature. At the same time, the model helps to prevent a common misunderstanding. The lesson is not that there literally exists a substantival "money space" inhabited by monetary entities. Rather, as argued earlier in this thesis, the moral of such analogies is that gauge formalisms capture a general structural feature: local state spaces or local conventions together with principled rules of comparison naturally give rise to geometric invariants.

Within this toy model, the natural structural reading is that what is physically significant is not the bare local monetary labels or the isolated exchange-rate entries taken one by one, but the invariant pattern of comparison they define across the market. In bundle language, the countries play the role of the base space, while the local currencies play the role of fibrewise frames or units in which value is expressed. The "bookkeepers" then play the role of a connection: they encode how one passes from one local monetary frame to a neighbouring one by specifying the exchange rates. This matches the general thesis defended in this work, according to which gauge theories describe objective relations of principled comparison and transport between local value spaces, with physical states given by the resulting transport structure up to gauge isomorphism rather than by primitive identities of fibre elements.

This becomes especially clear once one asks which quantities survive changes of local convention. Schwichtenberg insists that if a country rescales its currency, the exchange rates must be adjusted accordingly, and that the trader's net gain is unchanged by such a local rescaling. The point of the example is precisely that arbitrage does not depend on arbitrary choices of local units. In the mathematical formulation, the gauge-variant exchange rates are written in terms of variables $A_i(\vec{n})$, while the gauge-invariant content is encoded by loop quantities such as the arbitrage gain and its curvature-like logarithm $F_{ij}$. From the present perspective, this is the decisive structural lesson of the model: the physically meaningful content lies not in any privileged local representative, but in the comparison structure that remains invariant under local rescalings.

A first philosophical consequence concerns the status of passive transformations. In the toy model, a passive transformation is a mere change of description, such as replacing the lira by a rescaled lira. Schwichtenberg is explicit that such a transformation is only a change of the "local money coordinate system" and therefore should not have any physical effect, provided the bookkeepers adjust the exchange rates consistently. In the present idiom, passive transformations amount to changes of local frame or local trivialization: they alter the coordinate presentation of the local monetary values, but they leave the underlying transport structure untouched. They are therefore naturally interpreted as genuine representational redundancies. This is the equivalent of the frame transformations on the surface that we identified as similar to gauge transformations in Sec. 13.3. The choice to work with associated vector bundles as tools to represent matter fields already suppresses such redundancy by considering only equivalence classes.

The case of active transformations is subtler. In this context, vertical bundle automorphisms, which we identified as a symmetry of gauge theories, would represent a local change of money in a given currency, with a corresponding change in the bookkeepers that preserve transport structure. For the case of the geometry of the surface (Sec. 13.3) this would represent an active change of the vector's orientation in a given frame, with a corresponding change in the connection that "absorb" this change. Sophistications implies that we treat these situations as physically equivalent. On such a reading, an active transformation plus the appropriate transformation of the bookkeepers can be taken to define not a distinct physical market, but a different representative of the same invariant comparison structure. The difference between active and passive transformations then remains real, but it is representational rather than ontological: passive transformations change only the local description, whereas active ones change the representative configuration itself, even though both can remain physically idle when the same transport structure is preserved.

This also clarifies the metaphysical issue of identity. If two money-market descriptions differ only by an active gauge transformation, then they should count as describing the same physical situation through sophistication. This means that the local money-units and local exchange-rate entries cannot be fundamental all by themselves. There is no extra fact that one gauge-related local value is the uniquely real one. In that sense, they lack primitive haecceity: they do not have a built-in "this very one" identity across equivalent descriptions. And in a stronger sense, they also lack primitive quiddity: what they are is not fixed by an intrinsic essence independent of the rest of the model, but by the role they play in the overall structure of exchange and arbitrage relations. This does not eliminate local values. It only means that they are not basic independently of the transport structure that relates them. What is basic is the invariant comparison structure itself.

The toy model therefore supports a non-eliminativist structuralism. It does not imply that money and bookkeepers are unreal. It implies only that their local gauge-variant presentations are not fundamental. Money, in the model, is best understood not as a primitive substance, nor as an extra dimension of some space, but as a local unit of account whose representative form may vary without physical consequence. Likewise, the bookkeepers are real not as primitively individuated local entries, but as the bearers of a connection-like structure governing how values are compared and transported from site to site. The ontological core is thus the relational organization of exchange: which local values can be compared, how they compose along paths, and whether a closed loop yields a non-trivial invariant gain. This is precisely the sort of comparison-and-transport ontology proposed in the general case of gauge theory.

In this sense, Schwichtenberg's toy model offers a useful analogue for the broader thesis of this work. It shows, in a setting stripped of many physical complications, how gauge-variant local representatives can fail to be ontologically basic while invariant transport structure remains fully real and explanatory. At the same time, the example must be handled with care. Because money in the toy model already functions largely as a conventional local unit, the step from "local labels are conventional" to "their identity is structurally fixed" is easier to accept here than in Yang–Mills theory involving fields associated to fundamental particles. For example, in the case of QCD, money and currencies should be replaced by definitions of colors and amplitude of the wave-functions for each color. Bookkeepers should be replaced by gluon fields, which encode how to identify an arbitrary choice of color from one point of space-time to another (identically, with zero of the phase (real axis) for electrodynamics and weak isospin for the weak force). The example is therefore illuminating, but also somewhat flattering: it supports the plausibility of transport-structure realism, yet does not by itself settle the harder metaphysical questions in the physical gauge-theory case. The safest conclusion is accordingly modest: Schwichtenberg's model does not prove transport-structure realism, but it does provide a clean and pedagogically powerful instance of the kind of structural interpretation defended in this thesis.

# 19 Can such an interpretation be applied to the tangent bundle and velocities?

Our discussion might suggest that the relationalist or sophisticated package can indeed be extended to spacetime geometry, but only if one keeps distinct two issues that are often run together. The first concerns the status of spacetime points themselves. Here the analogy with gauge theory is direct: general relativity faces its own version of generalized-hole-style underdetermination, since diffeomorphism-related models may be mathematically distinct while agreeing on all physical facts. The familiar sophisticated response is therefore to treat diffeomorphisms—or, more precisely, isometries once a metric is present—as establishing an isomorphism between models rather than a difference in physical reality. In this sense, general relativity supports the same anti-haecceitistic lesson as gauge theory: one need not regard the primitive identity of manifold points as physically fundamental, but may instead be realist about the geometric structure—metric, connection, and curvature—only up to the relevant notion of isomorphism.

The second issue concerns tangent vectors such as velocities. Here too a transport-centred reading is appropriate, because the same locality constraint holds as in gauge theory: without further structure there is no canonical way to identify a vector in $T_xM$ with a vector in $T_yM$. To compare vectors at different spacetime points, one needs a rule of comparison, and in general relativity this rule is supplied by an affine connection, usually the Levi–Civita connection. Moreover, in curved spacetime the result of such comparison is generically path-dependent: parallel transporting the same vector along different curves can lead to different results, and this failure of path-independence is precisely what curvature measures. In that precise sense, the transport-structure moral applies to velocities just as it does to internal gauge quantities: cross-point comparison is not primitive, but fixed by the connection and the resulting transport structure.

However, the analogy should not be overstated. Although the tangent bundle can be treated mathematically as an associated vector bundle, tangent vectors are not "internal" in the same strong sense as Yang–Mills values. A four-velocity is already an intrinsic geometric object: it is simply the tangent vector to a timelike worldline. Changing coordinates or changing a local basis alters only the components used to represent that vector, not the vector itself. More importantly, the tangent bundle is canonically anchored to spacetime through the solder relation, which ties frame components to actual tangent vectors on the manifold. Because of this anchoring, tangent vectors have an immediate spacetime meaning—they represent directions, and once a metric is fixed they also have a norm and a causal character—whereas generic internal gauge vectors do not. For this reason, velocities are not "internally relative" in the strongest sense of Catren 2008. They are frame-relative in their representation and relational in their comparison across points, but they remain intrinsic geometric objects at each point.

This is also why one must be careful with active and passive transformations in the spacetime case. A passive transformation, such as a change of coordinates or a change of local frame, merely changes the representation of a fixed geometric situation. An active transformation can also be treated sophistication-wise as physically idle, but only when the whole spacetime model is transformed by an appropriate isomorphism, so that all the relevant geometric relations are preserved, not only vertical automorphisms. If one merely rotates a tangent vector while holding the spacetime setting fixed, one has usually changed an actual spacetime direction, not merely a gauge representative. By contrast, if spacetime, the vector field, and the connection are transformed together by the appropriate diffeomorphism or isomorphism, then the sophisticated identification becomes legitimate: what is preserved is the relational structure itself, not the primitive identity of points or vectors. This point relates strongly with our previous discussion in Sec. 14.9.

Rovelli's coupling moral also transfers only in a partial way. There is a weak analogue, because the measured three-velocity of a body is always relative to an observer's frame and is therefore relational in an operational sense. But the four-velocity itself does not owe its existence to such a coupling with another system. It is already defined as geometric data on spacetime; what depends on an observer is only its projection or interpretation as a three-velocity. This again marks a difference from internal gauge variables, whose physical interpretation is more deeply tied to the comparison structure itself.

The best conclusion is therefore a careful one. A transport-centred and sophisticated interpretation does extend naturally to spacetime geometry and to the tangent bundle: both gauge theory and general relativity support a non-haecceitistic reading in which physically salient content is given by structure up to isomorphism, and in both cases comparison across fibres is governed by connection-dependent transport rather than by primitive trans-point identity. But the extension is weaker in the spacetime case than in the internal gauge case. Internal gauge values live in fibres that are not canonically tied to spacetime, whereas tangent vectors are already genuine spacetime objects. The right claim is therefore not that velocities are "internally relative" in exactly the same sense as gauge values, but that they are intrinsically geometric objects whose cross-point comparison is irreducibly relational and transport-dependent.

The situation might however be subtler for general relativity on the spin-bundle (Appendix B). Spinors are not intrinsic geometric objects in the same way that tangent vectors are, because they are not associated directly with a spacetime orientation. As such, they are more similar to internal gauge values, and the transport-structure moral may apply to them in a stronger way. In such a setting, the differences between gauge theories and general relativity is also further reduced.

## 20 Going further: topology and quantum fields

### 20.1 Global topology and gauge

The discussion so far has been largely local. We treated gauge theories as theories of comparison and transport: a connection defines covariant derivatives, parallel transport along curves, and holonomies around loops. This already captures much of what is geometric about gauge theory. But it remains incomplete. Gauge theories also possess genuinely *global* features, encoded not merely in the local curvature at a point, but in the topology of the bundle and in the global homotopy classes of gauge-field configurations. These structures show that gauge theory is not only differential geometry; it is also topology.

The first indication comes from characteristic classes (for a presentation, see Chap. 11 of Nakahara (2003)). Even when a bundle is locally trivial[106], it may fail to be globally trivial, and this obstruction is measured by topological invariants such as Chern classes. In the abelian case, the first Chern class measures the global twisting of a U(1) bundle; physically, its integral gives the quantized magnetic flux of a monopole configuration. In non-abelian theories, higher characteristic classes appear, most notably the second Chern class (or equivalently the Chern–Pontryagin number in four dimensions), which classifies topological sectors of Yang–Mills fields. Such quantities are striking because they are gauge-invariant, integer-valued, and insensitive to smooth local deformations: they reveal that two gauge fields may satisfy the same local equations and yet belong to different global sectors.

Homotopy groups provide a complementary language for the same phenomenon. In many physically relevant situations, topological sectors are classified by maps from spheres into the gauge group or the vacuum manifold. Dirac's magnetic monopole already shows that a gauge potential may be definable only patchwise, with consistency on overlaps forcing magnetic charge quantization. In spontaneously broken non-abelian gauge theories, finite-energy monopoles are classified by the topology of the vacuum manifold, typically through $\pi_2(G/H)$; this is the framework in which the 't Hooft–Polyakov monopole appears ('t Hooft 1974; Polyakov 1974). Likewise, instantons in four-dimensional Euclidean Yang–Mills theory are classified by an integer winding number associated with $\pi_3(G)$. The moral is clear: gauge redundancy does not erase topology. On the contrary, once one quotients by small gauge[107] transformations, one often discovers a residual classification by large gauge transformations and homotopy classes.

These global sectors are not merely mathematical curiosities. They have dynamical and quantum significance. Instantons contribute non-perturbatively to the path integral by mediating tunnelling between classically distinct vacua. In non-abelian gauge theory, pure-gauge configurations with vanishing field strength need not all lie in the same connected component; instead, they fall into sectors labelled by an integer winding number. Quantum mechanically, the true vacuum may therefore be a superposition of these sectors, leading to the notion of $\theta$ vacua (Callan, Dashen, and Gross 1976; Jackiw and Rebbi 1976). The $\theta$ parameter multiplies a topological density, leaves the classical equations of motion unchanged, yet can have physical consequences, most famously in the strong CP problem (for a review, see Kim and Carosi 2010). This is an especially important lesson for interpretation: not all physically relevant structure in gauge theory is captured by local field strength or by classical local transport. Some of it lies in the global organization of the space of gauge-equivalence classes itself.

This topological perspective also sharpens the philosophical lesson of the Aharonov–Bohm effect discussed earlier. There, the salient structure was already not an intrinsic field value at a point but a global phase picked up around a loop. Berry 1984's phase generalizes this idea: when a quantum system is transported adiabatically around a closed circuit in parameter space, its state acquires a geometric phase depending on the path rather than only on local dynamical data. Berry showed that this phase can be understood in terms of a connection on a parameter-space bundle, with an associated curvature and holonomy. In this respect, geometric phase phenomena provide a bridge between gauge theory and a much broader class of physical systems: they display the same basic architecture of local gauge choice, connection, curvature, and globally meaningful holonomy.

Condensed-matter physics has made this bridge especially fertile. In band theory, Bloch states over

[106] See Appendix D.2 for a definition of trivial.

[107] Small gauge transformations can be continuously deformed to the identity. Large gauge transformations cannot.

the Brillouin zone define a natural Berry connection in momentum space, and its curvature controls measurable properties of materials. The integer quantum Hall effect provided the paradigmatic case: the Hall conductance is quantized because it is given by a topological invariant, the first Chern number of the occupied bands over the Brillouin torus (Thouless et al. 1982; Xiao, Chang, and Niu 2010). More broadly, Berry curvature and Chern numbers now organize the theory of topological insulators, anomalous Hall effects, orbital magnetization, and quantum pumping. Here again, gauge-theoretic language is not ornamental. It expresses the fact that one cannot in general choose a globally smooth phase convention for the quantum states; the obstruction is topological, and the observable consequences are robust precisely because they depend on global invariants rather than on local descriptive choices.

What this adds to the present essay is a final caution against an overly local reading of gauge geometry. If the core of gauge theory is, as argued above, a structure of comparison and transport, then topology shows that this structure must be understood globally as well as locally. Curvature at a point, holonomy around a loop, characteristic classes of bundles, and homotopy classes of field configurations are not competing ontologies; they are different levels at which the same gauge-geometric organization becomes visible. The geometric content of gauge theories therefore exceeds both naïve realism about local potentials and strict reduction to local field strengths. It includes the global topology of the bundle and of configuration space, whose empirical significance becomes especially vivid in monopoles, instantons, $\theta$ vacua, and Berry-phase phenomena.

## 20.2 Quantized gauge theories and symmetry breaking

Interpreting gauge theories is a crucial endavour located at the crossing point of all the most pressing questions of philosophy of physics. So far, we have largely neglected one of the most critical aspect of gauge theories: they are quantum theories, in which each element is a quantum field. While we discuss some quantum aspects of gauge in this essay, we glossed over most of its complexity.

At the quantum level, notions such as gauge fixing, redundancy, and observability acquire a new significance. The issue is no longer only which classical quantities should be regarded as physically real, but also how to understand the status of gauge-dependent variables in a theory whose empirical content is extracted through quantum states, operators, amplitudes, and path integrals. In that context, the tension already identified in the classical theory between local description and gauge equivalence becomes sharper rather than weaker. All these questions become intertwined with the challenge to interpret quantum field theories (Baker 2015; Kuhlmann 2023). In particular, one may ask how to interpret a transport structure when, at the quantum level, it is encoded in operators associated with particle creation and annihilation. If one takes seriously the contrast between particle and field ontologies, it becomes natural to ask whether the $\psi$ and $A$-fields might admit a unified interpretation in the quantum theory, despite the absence of any such common interpretation at the classical level. For example, if quantum field theory is taken to support a particle ontology, should matter particles (quarks, electrons, and so on) and force carriers (gluons, photons, and so on) be regarded as the same kind of entities, despite being associated with profoundly different geometrical structures at the classical level?

In quantum field theories, the procedure of gauge fixing, seems to be a necessary step in standard covariant quantization procedures (e.g. Gupta-Bleuler approach), questionning again the possibility of a preferred choice of gauge. Path integral quantization approaches also lead to the emergence of unphysical states, requiring the introduction of new unphysical degrees of freedoms, ghost fields, in order to preserve unitarity and gauge invariance in the quantized theory. Such complications leads some authors, such as Guay (2004) and Healey (2007), to prefer interpretations based on Wilson loops (C-type in Part II).

A major interpretative challenge is also given by symmetry breaking. In standard textbook presentations of the electroweak theory, the Higgs mechanism is often described as a spontaneous breaking of local gauge symmetry by a scalar field acquiring a non-zero vacuum expectation value. However, this way of speaking is at best heuristic. In the quantum theory, local gauge symmetry is usually understood not as an ordinary physical symmetry relating distinct physical possibilities, but as a redundancy of description. In lattice gauge theory, this point can even be made precise through Elitzur's theorem: local gauge symmetries are not spontaneously broken in the straightforward sense in which global symmetries may be. What may break instead are residual global symmetries that remain after gauge fixing, and these do not map in any simple way onto the genuinely physical content of the theory.

The philosophical consequence is important. If local gauge symmetry is not itself a physical symmetry of nature, then the empirical success of the Higgs mechanism cannot simply consist in nature breaking such a symmetry. Its physical significance must instead be sought in gauge-invariant features of the theory: in the restructuring of the spectrum, in the appearance of massive vector bosons, and in the organization of observable states. In that sense, quantized gauge theory reinforces one of the main lessons reached in the previous chapters: what matters physically is not a particular gauge-dependent representative, but the invariant structure carried by the theory.

Quantization also sharpens several other questions that have only been touched on here. The Aharonov-Bohm effect already suggested that non-local quantities such as holonomies can be physically significant; quantum Yang-Mills theories deepen this lesson through confinement, anomalies, and the non-perturbative importance of Wilson loops. Likewise, the use of path-integral techniques and other quantization methods complicates any naive realism about the fields appearing in the Lagrangian. A fully satisfactory interpretation of gauge theories must therefore go beyond the classical bundle picture and confront the quantum theory on its own terms. The BRST quantization procedure rooted in differential geometry, is probably the best direction to start such an investigation. For a discussion of this point, see e.g. Guay (2008), in which the author claims that gauge symmetry, and the gauge principle, might be significant exactly because it allows for quantization with the BRST procedure.

For these reasons, the present work should be read as an investigation of the classical and geometrical core of gauge theory, not as a complete philosophy of gauge theories as they function in contemporary fundamental physics. Quantized gauge theories and symmetry breaking do not invalidate the conclusions reached above, but they do require them to be reformulated more carefully. They suggest that the deepest lesson of gauge theory is not that the world contains hidden local symmetries that are sometimes broken, but rather that our most successful theories are organized around equivalence classes, gauge-invariant structures, and subtle relations between redundancy, locality, and observability.

Let us conclude this section with the following remark: while the mathematical and geometrical structures of classical theories such as gauge theory and general relativity are well established, the mathematical foundations of quantum field theory remain largely an open problem. As such, it is difficult to establish a robust interpretative framework on such uncertain grounds. The question of whether the geometrical interpretations developed in general relativity and gauge theory can be extended to the quantum realm largely depends on the mathematical tools that will be developed in the future to provide a rigorous definition of quantum field theory, as well as on the reconciliation of gravity with quantum theory. Furthermore, as we have seen multiple times in this essay, classical geometric differences between gauge and gravity are deeply linked to deep underlying quantum problems, and we can only hope, along with Weyl, Kaluza and so many others, that the geometrical foundations that underly gauge and gravity admit a deeper and more unified foundation. While a detailed follow-up work focusing on quantum gauge theories would be valuable to our project, future experimental and theoretical developments in physics might be the missing step allowing us to pursue this discussion further.

## 21 Conclusion

The aim of this work was to answer a question that is at once historical, conceptual, and metaphysical: what is geometric about gauge theories? To answer it, we followed the notion of potential from classical mechanics to electromagnetism, then from relativistic field theory to Yang–Mills theory, in order to understand how gauge structures progressively acquired their peculiar status in modern physics. This historical reconstruction, done in Part I, showed that gauge theories did not emerge as fully formed "geometric theories". Rather, they arose from a long displacement in the status of potentials: from auxiliary computational devices, to indispensable theoretical structures, and finally to objects whose physical significance could no longer be dismissed once the Aharonov–Bohm effect and the modern covariant formulation of field theory were taken seriously.

Part II then showed why the interpretation of gauge theories remains philosophically difficult. Gauge symmetry introduces a form of representational redundancy that threatens familiar ideas of determinism, locality, separability, and the identification of genuine physical quantities. The traditional opposition between field strengths, local gauge potentials, and holonomies captures part of the problem, but not all of it. In particular, the Aharonov–Bohm effect makes it impossible to remain satisfied with the simple classical view according to which only local field strengths are real, while at the same time it does not license a naïve realism about gauge potentials themselves. The interpretative problem is therefore not merely to choose between $F$, $A$, or holonomy, but to determine what kind of structure a gauge theory represents when its formalism contains both indispensable local machinery and manifest redundancy.

In Part III, we reviewed the geometric formulation of gauge theories. The fiber-bundle formalism provided the decisive conceptual tool to allow for such a reformulation. In that framework, gauge theories can indeed be understood geometrically: matter fields are represented as sections of associated bundles, gauge fields as local representatives of a connection, and field strengths as curvature. On this reading, gauge invariance expresses the fact that the physical content of the theory does not depend on an arbitrary choice of local frame. Yet this geometric reformulation must be handled carefully. It would be too quick to conclude that gauge theories describe a literally existing arena of additional internal spaces populated by independently real geometrical objects. The formalism shows that gauge theories are geometric, but it does not by itself decide how that geometry should be interpreted.

The central claim defended here is that the most convincing lesson of the geometric formulation is not fiber-bundle substantivalism, but a more modest and, I think, more robust structural realism. What is physically significant in gauge theories is not a collection of substantival fibres or primitive identities of fibre elements, but the structure of comparison and transport encoded by the connection, together with its curvature and holonomy content. Gauge theories are geometric in the sense that they organize local state spaces by means of principled rules of parallel transport, and because observable effects may depend on this transport structure even when no local force field is present. What is geometric, therefore, is not primarily the existence of an additional "space" alongside space-time, but the existence of an objective comparative structure linking local descriptions across space-time.

This position also clarifies the relation between gauge theories and general relativity. The analogy between the two is deep and should not be dismissed as a mere formal curiosity: both theories rely on connections, curvature, holonomy, locality, and redundancy of description. At the same time, the analogy has limits. In Yang–Mills theory, the relevant geometry concerns internal spaces attached to space-time, whereas general relativity is concerns about vectors associated to specific directions in space-time such that the geometry at stake is that of space-time itself. Furthermore, the two theories differ importantly in their dynamics, since the Lagrangian density of Yang–Mills theory is quadratic in curvature whereas the one of general relativity is linear. These differences prevent any crude assimilation of gravity to an ordinary gauge theory, but they do not undermine the broader lesson that both should be approached within a common interpretative framework sensitive to their shared geometric structures. Physics beyond the general relativity and the standard model of particle physics may further bridge these gaps.

The answer to the title question can therefore be stated as follows: gauge theories are geometric because they describe how local internal states are related, compared, and transported, not because they necessarily posit additional substantival spaces over and above space-time. Their genuine physical core lies in a transport structure defined only up to gauge isomorphism. This explains both why gauge symmetry is not itself an interaction-generating principle and why the language of differential geometry is nevertheless indispensable to the formulation and interpretation of gauge theory. Invariance under change of frame is physically empty taken by itself; what matters is the non-trivial connection and curvature structure that constrains matter and gives rise to observable holonomy effects.

This conclusion remains provisional in an important sense. As the previous chapter emphasized, the present essay has been restricted essentially to classical gauge theories, while the physically fundamental theories are quantum field theories. Quantization, non-perturbative observables, confinement, anomalies, and symmetry breaking may all reshape the interpretative landscape. The Higgs mechanism in particular, as well as the status of gauge symmetry in the quantized theory, must be integrated into any fully satisfactory account. The framework defended here should therefore be understood not as a final ontology of all gauge physics, but as a proposal for the classical and semi-classical core from which a more complete interpretation might proceed.

If one wanted to condense the overall result of this mémoire into a single formula, it would be this: gauge theories are not best understood as theories of hidden local properties, nor merely as catalogues of gauge-invariant observables, but as theories of structured comparison. Their geometry is the geometry of local state spaces plus rules of transport between them. In that sense, the philosophical importance of gauge theories is not only that they challenge old metaphysical categories, but that they force us to rethink what it means for a physical theory to be geometric at all.

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# Appendix

## A Basic vector operations and the $\nabla$ operator

We recall that, vectors are arrows which components can be expressed in a given basis. Considering two vectors $\vec{A}$ and $\vec{B}$, we can define the following operations:

- The *dot product* $\vec{A} \cdot \vec{B}$ is a scalar defined as $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$ where $A_x, A_y, A_z$ and $B_x, B_y, B_z$ are the components of $\vec{A}$ and $\vec{B}$ in a given basis. It is a measure of the "parallelism" between the two vectors, being zero if they are perpendicular and positive (resp. negative) if they are more parallel in the same (resp. opposite) direction.

- The *cross product* $\vec{A} \times \vec{B}$ is a vector defined as $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\vec{e}_x + (A_z B_x - A_x B_z)\vec{e}_y + (A_x B_y - A_y B_x)\vec{e}_z$ where $\vec{e}_x, \vec{e}_y, \vec{e}_z$ are the unit vectors of the basis. It is a measure of the "orthogonality" between the two vectors, being zero if they are parallel and having a magnitude proportional to the product of the magnitudes of the two vectors and the sine of the angle between them, and a direction given by the right-hand rule.

In Cartesian coordinates $(x, y, z)$ with associated unit vectors $\vec{e}_x, \vec{e}_y, \vec{e}_z$, the $\vec{\nabla}$ operator can be expressed as

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{e}_x + \frac{\partial}{\partial y}\vec{e}_y + \frac{\partial}{\partial z}\vec{e}_z \tag{63}$$

It can act on a function $f(x, y, z)$ to give it's *gradient*

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{e}_x + \frac{\partial f}{\partial y}\vec{e}_y + \frac{\partial f}{\partial z}\vec{e}_z \tag{64}$$

which is a vector field which gives the amplitude and direction of change of $f$ in every point of space. $\vec{\nabla}$ can also act on a vector field $\vec{V}(x, y, z) = V_x(x, y, z)\vec{e}_x + V_y(x, y, z)\vec{e}_y + V_z(x, y, z)\vec{e}_z$ using the dot product to gives the divergence

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{65}$$

which is a function indicating how much of the vector field "goes in or out" in every point of space. Finally $\vec{\nabla}$ can also acts on vector fields through the cross product $\times$ to give it's curl

$$\vec{\nabla} \times \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{e}_z \tag{66}$$

which is another vector field carrying information about the magnitude and axis of an induced rotation of the vector field on an infinitesimal object if $\vec{V}$ was the velocity field of a fluid.

They obey the following relations $\forall f \in C^\infty(\mathbb{R}^3)$ and $\forall$ vector field $\vec{V}$:

$$\vec{\nabla} \times \vec{\nabla} f = 0 \tag{67}$$

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{V} = 0 \tag{68}$$

## B General relativity in a nutshell

### B.1 On vectors (tangent bundle)

Gauge theories as presented above describe all the fundamental interactions known between matter fields except gravitation. Gravitation is best described today by the theory of general relativity, in which interaction is commonly interpreted as the consequence of the geometry of space-time[108]. As we would like to discuss the geometrical interpretations of gauge theory, let us give a presentation of general relativity highlighting as much as possible the similarity and differences between gauge

[108] The are several ways in which this geometrical content can be re-interpreted and it is even possible to reject the geometrical content of general relativity. In order to avoid blurring our discussion, we will not address these points here.

and gravity. As for gauge theories, our presentation here will be dense and assume that the reader is already familiar with the topic. Excellent and complete presentations of general relativity can be found in Misner, Thorne, and Wheeler (1973), Reula (2010), Wald (1984), and Weinberg (1972).

In general relativity, space-time is a four dimensional manifold $M$. $M$ is equipped with a metric tensor field $g$ of signature $(-,+,+,+)$ allowing to define the length and angles between vectors in each tangent space of $M$. Derivation of vectors on $M$ is performed by an affine connection $\nabla$. Tangent vector spaces at different points of $M$ are entirely independent entities and $\nabla$ encodes how to connect and compare them, by allowing for the notion of parallel transports of vectors over $M$. Affine connections are defined in full generality in Appendix D.3.

Choosing a specific frame field $e_\mu = \partial_\mu$ over a chart $U \subseteq M$, the covariant derivative of the frame is given by a set of coefficients $\Gamma^\lambda_{\ \mu\nu}$ known as the connection coefficients or **Christoffel symbols** and defined as[109]

$$\nabla_\mu e_\nu = \Gamma^\lambda_{\ \mu\nu} e_\lambda, \tag{69}$$

such that the covariant derivative of any vector $v = v^\mu e_\mu$ can be expanded as

$$\nabla_\nu v = \left(\partial_\nu v^\mu + \Gamma^\mu_{\ \nu\lambda} v^\lambda\right) e_\mu. \tag{70}$$

One of the fundamental assumption of general relativity is that the affine connection $\nabla$ is the unique **Levi-Civita connection** which is both torsion free and metric preserving (see Appendix D.3). As such its coefficients can be written in terms of the metric tensor as

$$\Gamma^\lambda_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}). \tag{71}$$

This choice is made both from simplicity arguments and in order to fulfill the so-called Einstein equivalence principle[110] (for a definition see Will (2014)). This principle requires the possibility to find everywhere an inertial free falling frame equivalent to Minkowski space-time i.e. there exists a transformation of the metric $g \to \eta$ in all tangent space. We say that $M$ is locally Minkowskian. Moreover, the Einstein equivalence principle requires that all point like particle of mass $m$ and velocity $u^\mu$ follow the same motion on $M$ given by the connection as

$$\nabla_u u = 0. \tag{72}$$

We say that the particle follows **geodesic** motion, which corresponds to the path which extremalize the space-time length given by $g$.

As detailed in appendix D.4, the curvature of a linear connection is quantified by the **Riemann tensor**

$$R^\lambda_{\ \mu\rho\nu} = \partial_\rho \Gamma^\lambda_{\ \mu\nu} - \partial_\nu \Gamma^\lambda_{\ \mu\rho} + \Gamma^\sigma_{\ \nu\mu}\Gamma^\lambda_{\ \rho\sigma} - \Gamma^\sigma_{\ \rho\mu}\Gamma^\lambda_{\ \nu\sigma} \tag{73}$$

which would then be entirely determined by the metric $g$. $R^\lambda_{\ \mu\rho\nu}$ quantifies the deviation between nearby geodesics, which is interpreted as tidal forces. As such, the metric $g$ describing the geometry of vectors in each tangent space also defines the connection describing the transport and variation of vectors over space-time. The curvature of this connection, also given solely in terms of the metric $g$, can thus be interpreted as the curvature of the space-time $M$. As such, all the key elements of general relativity concern the space-time geometry through $g$.

Let $\mathcal{L}_m$ be any Lagrangian describing the presence of matter[111], $R_{\mu\nu} = R^\lambda_{\ \mu\lambda\nu}$ and $R = R^\mu_{\ \mu}$. General relativity is the theory defined by the Lagrangian density

$$\mathcal{L}_{\mathrm{GR}} = \mathcal{L}_m + \frac{R}{16\pi G}. \tag{74}$$

[109] Noting $\nabla_\mu = \nabla_{e_\mu}$.

[110] Extensions of general relativity allowing for a non vanishing torsion of the connection have been proposed, as the so-called "Einstein-Cartan" theory. For a review, see e.g. Trautman (2006).

[111] $\mathcal{L}_m$ should be identical to the Lagrangian density used to describe matter fields in flat Minkowski space-time in which the metric $\eta$ is replaced by the metric $g$.

This Lagrangian density leads the geodesic equation and the **Einstein equations** describing the interrelation between matter and space-time geometry

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{75}$$

where the stress-energy tensor of the matter component is defined as $T_{\mu\nu} = -2\partial\mathcal{L}_m/\partial g^{\mu\nu} + g_{\mu\nu}\mathcal{L}_m$.

## B.2 On matter fields (spin bundle)

Quite interestingly, it is possible to define general relativity acting on matter fields $\psi$ in a similar fashion as how we defined gauge theories in Sec. 8.2. To do so, we follow the proposition of Donoghue (2017) based on ideas already presented by Utiyama (1955) and Kibble (1961) to put forward the similarities between Yang-Mills theory and general relativity.

Let $\psi$ be a matter field. As a spinor, it lives in irreducible representations of the Lorentz group and can only be defined in a four dimensional vector-space equipped with a Minkowski metric. Luckily, as space-time $M$ is locally Minkowskian, there must exist a frame transformation in each tangent space going from the natural basis $e_\mu = \partial_\mu$ to a basis $\tilde{e}_I$ which is Minkowskian for $g$ i.e. in which $g(\tilde{e}_I, \tilde{e}_J) = \eta_{IJ}$. We write $e^I_{\ \mu}$ and $e^\mu_{\ I}$ the transformation matrices going from one frame to another as $e_\mu = e^I_{\ \mu}\tilde{e}_I$ and $\tilde{e}_I = e^\mu_{\ I}e_\mu$. Knowing these transformations at each point of space-time is equivalent to knowing the metric as

$$g_{\mu\nu} = g(e_\mu, e_\nu) = g(e^I_{\ \mu}\tilde{e}_I, e^J_{\ \nu}\tilde{e}_J) = e^I_{\ \mu}e^J_{\ \nu}g(\tilde{e}_I, \tilde{e}_J) = e^I_{\ \mu}e^J_{\ \nu}\eta_{IJ} \tag{76}$$

Let now $G$ =SO(1,3) be the Lorentz group and $\rho(G)$ be its SPIN representation (SPIN(1,3)$\simeq SL(2,\mathbb{C})$). We can write introduce the spin covariant derivative defined as

$$\widehat{\nabla}_\mu = \partial_\mu + \frac{1}{4}\tilde{\omega}^{IJ}_\mu \Sigma_{IJ}. \tag{77}$$

$\Sigma_{IJ}$ are the the generators of the Lorentz transformations in the spinorial representation which can be obtained from the gamma matrices as $\Sigma_{IJ} = \frac{1}{4}(\gamma_I\gamma_J - \gamma_J\gamma_I)$. The term $\tilde{\omega}^{IJ}_\mu$ is called the spin-connection and is directly related to the Levi-Civita connection as

$$\tilde{\omega}^{IJ}_\mu = e^I_\nu \Gamma^\nu_{\sigma\mu} e^{\sigma J} + e^I_\nu \partial_\mu e^{\nu J} \tag{78}$$

From $\tilde{\omega}^{IJ}_\mu$, one can derive the curvature of the spin connection as

$$F^{IJ}_{\mu\nu} = \partial_\mu \tilde{\omega}^{IJ}_\nu - \partial_\nu \tilde{\omega}^{IJ}_\mu + f^{IJ}_{KLMN}\tilde{\omega}^{KL}_\mu \tilde{\omega}^{MN}_\nu \tag{79}$$

where $f^{IJ}_{KLMN}$ are the group structure constants of SO(1,3).

General relativity for a relativistic matter field is then defined by the Lagrangian

$$\mathcal{L}_{g\psi} = \overline{\psi}(\mathrm{i}\gamma^\mu \widehat{\nabla}_\mu - m)\psi - \frac{1}{4\kappa^2}e^\mu_I e^\nu_J F^{IJ}_{\mu\nu} \tag{80}$$

where $\gamma^\mu$ is obtained from the flat Dirac matrix $\gamma^I$ as $\gamma^\mu = e^\mu_I \gamma^I$. After extremalization, this Lagrangian gives back the Dirac equation with the covariant derivative $\widehat{\nabla}_\mu$ for $\psi$ and equations equivalent to the Einstein equations for $F^{IJ}_{\mu\nu}$. We introduced here $\kappa = \sqrt{4\pi G}$ which behaves as a universal gauge coupling for gravity. Note that, in the case of Yang-Mills theories, the gauge coupling $\overline{g}$ can be absorbed in the definition of the couplings $A^\alpha_\mu$, and reappears in the Lagrangian as $1/(16\pi\alpha_{\overline{g}})$, with $\alpha_{\overline{g}} = \overline{g}^2/(4\pi)$ is the so-called gauge fine-structure constant. As such, $G$ acts for gravity as some kind of fine-structure constant. See our discussion in Sec. 14.6.

Let $S = e^{\Sigma_{IJ}\theta^{IJ}}$ be a general Lorentz transformation. The theory is invariant under the gauge transformations

$$\begin{cases} \psi \to \psi' = S(x)\psi, \\ \tilde{\omega}^{IJ}_\mu \Sigma_{IJ} \to (\tilde{\omega}^{IJ}_\mu)'\Sigma_{IJ} = S(\tilde{\omega}^{IJ}_\mu \Sigma_{IJ})S^{-1} - (\partial_\mu S)S^{-1}. \end{cases} \tag{81}$$

As such, it is very tempting to identify the above formulation of GR as a gauge theory of the Lorentz group, identified as the group of transformations relating different inertial frames.

# C Mathematical and philosophical glossary

## C.1 Mathematics

- **isomorphism (category theory definition)**: Let $A$, $B$ two objects. A map $f : A \to B$ is an isomorphism if there exist $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.
- **homeomorphism**: A map $f : A \to B$ between two topological spaces $A$ and $B$ is called an homeomorphism if it is bijective ($\forall x \in A, !\exists y \in B$ such that $f(x) = y$), continuous ($\forall U \subseteq B$, $f^{-1}(U) = \{x \in A$ such that $f(x) \in U\}$) and if the inverse map $f^{-1}$ is also continuous. Homeomorphisms are corresponds to isomorphisms on (the category of) topological spaces.
- **isomorphism of algebraic structures**: Let $A$ and $B$ be two algebraic structures equipped with at least one internal operation noted $*_A$ and $*_B$ respectively. Let $f : A \to B$ be a map. $f$ is an isomorphism if it is both bijective ($\forall x \in A, !\exists y \in B$ such that $f(x) = y$) and structure preserving (homomorphism : $\forall x_1, x_2 \in A$, $f(x_1 *_A x_2) = f(x_1) *_B f(x_2)$). Two structures $A$ and $B$ are isomorphic if there exists at least one isomorphism between them. This corresponds to the general definition of isomorphism from category theory (above) on the special case of categories of algebraic structures.
- **pullback and pushforward**: Let $\phi : M \to N$ be a smooth map between two manifolds $M$ and $N$. Let $f \in C^\infty(N)$ be a smooth function $f : N \to \mathbb{R}$. The pullback of $f$ by $\phi$, noted $\phi^* f$, is the function $\phi^* f : M \to \mathbb{R}$ defined as $\phi^* f = f \circ \phi$. The pushforward allows to transform a vector field $v \in \Gamma(TM)$ to a vector $\phi_* v \in TN$ defined as $\phi_* v(f) = v(\phi^* f)\, \forall f \in C^\infty(M)$. The pullback of $\phi$ can be used to transform a 1-form (covector) $\omega$ on $TN^*$ in a 1 form $\phi^*\omega$ on $TM^*$ as $(\phi^*\omega)v = \omega(\phi_* v)\, \forall v \in TN$.
- **surjective map:** A map $f : A \to B$ is surjective if $\forall y \in B$, $\exists x \in A$ such that $y = f(x)$.

## C.2 Metaphysics

- **Reification**: Treating something introduced as a *tool of description* (a mathematical object, a convenient variable...) as if it were *literally a part of the world*. In other words, we *reify* a theoretical entity when we take "it appears in the formalism" to imply "it exists in reality".
- **Haecceitism**: "the doctrine that objects have an intrinsic identity (or 'thisness': haecceitas); haecceitistic possibilities involve individuals being "swapped" or "exchanged" without any qualitative difference;" (taken from Gomes 2022).
- **Quidditism**: "the doctrine that objects have intrinsic properties; and that properties can be "swapped" or "exchanged" without any qualitative difference;" (taken from Gomes 2022).
- **Principle of identity of indiscernible (PII)**: A metaphysical principle stating that if two entities share *all* their properties (or all their qualitative properties, depending on the version), then they are not two entities at all but one and the same. Formally: if for all properties $P$, $P(a) \leftrightarrow P(b)$, then $a = b$. (Some philosophers restrict the quantification to *qualitative* properties to avoid trivial counterexamples involving "being identical to $a$".).
- **Universals**: A traditional realist view about properties. A *universal* is a *repeatable* entity that can be wholly present in many places at once. For example, on this view, the redness of a red apple and the redness of a red book are instances of *one and the same* universal *Redness*. Universals are introduced to explain objective resemblance: two things resemble because they literally share a universal.
- **Tropes**: A rival realist view about properties. A *trope* is a *particularized* property-instance: the apple's redness is *one individual entity*, and the book's redness is *another*. The two tropes can be exactly similar, but they are numerically distinct. Tropes aim to explain resemblance without positing repeatable universals: two objects resemble because their tropes are sufficiently similar (often: exactly resembling).

- **Natural (sparse) properties**: Not a special *kind* of property like universals or tropes, but a distinction about *which* properties matter. Lewis 1986 notes that there are endlessly many "properties" in the loose sense (any condition you can write down, e.g. "being an electron or the Eiffel Tower"). These are *abundant* and do not track genuine similarity. By contrast, *natural* (or *sparse*) properties are the comparatively few properties that correspond to the real patterns and "joints" in nature: sharing them makes things objectively similar and they are the ones suited to figure in fundamental laws.

# D A primer on differential geometry

In this appendix, we define and explain all the essential mathematical concepts required to understand the core of our discussion. This presentation is not complete nor perfectly rigorous.

## D.1 Manifolds and their tangent bundles

The notion of smooth manifold generalizes the notion of surfaces to model continuous spaces in any arbitrary dimension. For a formal definition based on topological spaces see e.g. Nakahara 2003; Schuller 2015. Let $M$ be a manifold of dimension $d$. A **function** is a map $f : M \to \mathbb{R}$ and a **curve** is a map $\gamma : \mathbb{R} \to M$. The space of functions on $M$ is noted $C^\infty(M)$.

To each point $x$ of $M$, one can canonically associate a $d$ dimensional real vector space $T_xM$ known as the **tangent space**. For a Baez and Muniain 1994 and Marsh 2014. The combination of all the vector spaces is known as the **tangent bundle**. The choice of a vector $v \in T_xM$ is a **tangent vector** and the choice of a vector $v(x)$ in each tangent space is called a **vector field**.

A local choice of coordinates $x = (x^0, x^1 ... x^d)$ over a region $U \subseteq M$ is called a **chart**. Each $x^\mu : M \to \mathbb{R}$ is a function. In a chart, the derivative operator $\partial_\mu$ acting on functions form a faithful representation of the basis vectors $e_\mu$ of $T_xM$ known as a **natural basis**. As such, in $U$, a vector field can be decomposed as $v(x) = v^\mu(x)\partial_\mu$. As such a vector field is a map $v : C^\infty(M) \to C^\infty(M)$ acting on functions $v : f \to v(f) = v^\mu \partial_\mu f$.

The cotangent space $(T_xM)^*$ is the space of all maps $T_xM \to \mathbb{R}$ that is all maps transforming a vector into a real number. A general element of $(T_xM)^*$ is called a **covector**. The combination of all the cotangent spaces form the cotangent bundle $(TM)^*$. A **covector field** or a **one form** is a choice of covector in each cotangent space. As such, it is a map $TM \to C^\infty(M)$ taking a vector field and giving a function (i.e. a real number at each point of $M$). The **differential** $\mathrm{d}f \in (TM)^*$ of a function $f \in C^\infty(M)$ is a 1-form satisfying $\mathrm{d}f(v) = v(f)\ \forall v \in TM$. One can compute the differential of the coordinate function $\mathrm{d}x^\mu$ satisfying $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$. From it, it is possible to express any 1-form $\alpha \in (TM)^*$ in a chart $U$ as $\alpha = \alpha_\mu\, \mathrm{d}x^\mu$. $\alpha$ then act on any vector $v$ to give the real number $\alpha(v) = \alpha_\mu v^\mu$.

A rank $(m, n)$-tensor field is a map $\underbrace{TM^* \times ... \times TM^*}_{m \text{ times}} \times \underbrace{TM \times ... \times TM}_{n \text{ times}} \to C^\infty(M)$ which takes in input $m$ covector fields and $n$ vectors fields to give back a function. The space of tensor fields is noted $\mathcal{T}_{m,n}(M) = \underbrace{TM \otimes ... \otimes TM}_{m \text{ times}} \otimes \underbrace{TM^* \otimes ... \otimes TM^*}_{n \text{ times}}$. In a chart a tensor $T \in \mathcal{T}_{m,n}$can be decomposed as

$$T = T^{\mu...\nu}{}_{\rho...\sigma}\, \underbrace{\partial_\mu \otimes ... \otimes \partial_\nu}_{m \text{ times}} \otimes \underbrace{\mathrm{d}x^\rho \otimes ... \otimes \mathrm{d}x^\sigma}_{n \text{ times}} \tag{82}$$

where $\otimes$ is the **tensor product** defined for two covectors $\alpha$ and $\beta$ acting on two vectors $v$ and $w$ as $\alpha \otimes \beta(v, w) = \alpha(v)\beta(w)$. $T$ acts on $m$ covector fields $\alpha, ..., \beta$ and $n$ vector fields $v, ..., w$ to give a function as

$$T(\alpha, ..., \beta, v, ..., w) = T^{\mu...\nu}{}_{\rho...\sigma}\alpha_\mu ... \beta_\nu v^\rho ... w^\sigma \tag{83}$$

An example of tensor field is the **metric** $g \in T_{0,2}(M)$, which take two tangent vectors to return a real number in each tangent space i.e. $g : TM \times TM \to C^\infty(M)$. $g$ is symmetric $g(u, v) = g(v, u)$ and non degenerate (if $g(u, v) = 0\ \forall v \in TM$ then $u = 0$). In a chart, $g = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes\, \mathrm{d}x^\nu$ acting on two vectors $v, w$ as $g(v,w) = g_{\mu\nu}v^\mu w^\nu$. A manifold equipped with a metric tensor is called a **(pseudo)-Riemannian** manifold.

A $p$-form is an anti-symmetric tensor field of rank $(0,p)$. The space of $p$-form is written $\Omega^p(M)$. By construction, 0-forms are identified with functions $\Omega^0(M) = C^\infty(M)$ and 1-forms are the covectors $\Omega^1(M) = (TM)^*$. The exterior product $\wedge : (TM)^* \times (TM)^* \to \Omega^1(M)$ between two 1-forms $\alpha$ and $\beta$ is defined as $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$. In a local chart, the quantity $\underbrace{\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge ... \wedge \mathrm{d}x^\rho}_{p \text{ times}}$ forms a basis for the space $\Omega^p(M)$. The **exterior derivative** d is a map $\Omega^p(M) \to \Omega^{p+1}(M)$ which satisfies: i) $\mathrm{d}f(v) = v(f)$ if $f \in \Omega^0(M)$ ii) $\mathrm{d}(\alpha+\beta) = \mathrm{d}\alpha + \mathrm{d}\beta$ and $\mathrm{d}(k\alpha) = k\,\mathrm{d}\alpha$ for all $\alpha, \beta \in \Omega^p(M)$ and $k \in \mathbb{R}$ iii) $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$ for all $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$ and iv) $\mathrm{d}^2\alpha = 0$ $\forall \alpha \in \Omega^p(M)$.

The **Hodge star** $\star : \Omega^p(M) \to \Omega^{d-p}(M)$ is the map defined as $\alpha \wedge (\star\beta) = \langle \alpha, \beta \rangle_g \mu$ where $\langle \alpha, \beta \rangle_g = \alpha_{\mu...\nu}\beta^{\mu...\nu}$ and $\mu = e_1 \wedge ... \wedge e_d$ is the **volume element** in an orthonormal basis $e_\mu$ (satisfying $g(e_\mu, e_\nu) = \delta_{\mu\nu}$ in the Riemannian case or $g(e_\mu, e_\nu) = \eta_{\mu\nu}$ in the pseudo-Riemannian one.

## D.2 Fiber bundles

A **bundle** is a triplet $(E, M, \pi)$ where $E$ and $M$ are two smooth manifolds called respectively the **total space** and **base space** and $\pi$ is a surjective map $\pi : E \to M$ called the **projection map**. Let $x \in M$. The set $\pi^{-1}(x) = F_x$ is called the **fiber** over $x$. If the fibers are all homeomorphic to a typical fiber noted $F$, the bundle is said to be a **fiber bundle**. The typical visualization associated with a fiber bundle is that a copy of $F$ is "attached" at every point of the base space $M$. As such an (infinite) cylinder is a fiber bundle on which a fiber $F = \mathbb{R}$ has been attached at every point of a circle $M = S^1$.

Let $U \subseteq M$ be an open set of $M$. Fiber bundles are locally equivalent to the product space $\pi(U) \simeq U \times F$. If a bundle $E$ is equivalent to $F \times M$ over the whole base space, it is said to be **trivial**. A **section** $\sigma$ of a fiber bundle $E$ is a map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{Id}_M$. The space of the sections of a bundle $E$ is noted $\Gamma(\mathcal{E})$. Sections generalizes function by associating an element of $F$ at each point $x$ of $M$.

If the fibers are homeomorphic to a Lie group $G$, we talk about a **principal fiber bundle** or a $G$**-bundle**. $G$ is called the structure group of the principal fiber bundle. Locally, an element $p$ of $P$ can be written $p = (x, e)$ where $x$ is a point of the base space and $e$ a group element in $G$. $G$ has a natural (smooth and free) right action on the points $p \in P$ defined locally as $ph = (x, eh)$ for $h \in G$.

Let $P$ be a $G$-bundle over the base space $M$. Let $\rho$ be a representation of $G$ on a vector space $V$. Let $(p, s) \in P \times V$. We define the following equivalence relation:

$$(p, s) \sim (ph, \rho(h^{-1})s) \qquad \forall h \in G. \tag{84}$$

The **associated (vector) bundle** $\mathcal{E}$ to $P$ through the (left) action of $G$ on $V$ is defined as $\mathcal{E} = (P \times F)/\sim$ also noted $P \times_\rho F$, with the projection $\pi_\mathcal{E} : \mathcal{E} \to M$. Locally, an element of $\mathcal{E}$ is given by the equivalence class $[p, s]$. $\pi_\mathcal{E}$ is defined from the projection $\pi$ of $P$ as $\pi_\mathcal{E}([p, s]) = \pi(p)$. The principal bundle $P$ can be understood as the bundle associating at each point $x$ of the base space $M$ all the possible frames of the vector space $V_x = \pi_\mathcal{E}^{-1}(x)$. The correspondence between the element $p \in P$ and a frame $\hat{e}_i \in V$ can only be done once an arbitrary frame is identified with the unit element of the group, which is the case when a section is chosen locally. Indeed, as long as no identity section is chosen, the fiber cannot be identified with frames because there is no way to associate a frame with the identity element. Such a structure is called a "G-torsor"). All the frames of $P$ are related to one another by transformations of $G$. If $G$ =GL$(d)$, the frames can be any frame related to one another by general transformations. The choice of any other group, translates the restriction to a specific class of frames in order to preserve a specific structure. $G$ =SO$(d)$ indicates the restriction to orthonormal frames associated to the conservation of the inner product given by a metric $g$ as well as a specific orientation while the restriction to $G$ =SU$(d)$ is the sign of a restriction to complex orthonormal frames associated to the conservation of an hermitian product and an orientation. The element $p = (x, e) \in P$ can be identified with a frame $\hat{e}_i$ of $V$ at the point $x$. As such, we can write a general section $s \in \Gamma(\mathcal{E})$ in the form $s = s^i(x)e_i(x)$ where $\{e_i(x)\} = [p, \{\hat{e}_i\}] \in \mathcal{E}$ is the frame section in the associated bundle identifying the basis in $V$ and the element $p \in P$.

The **tangent bundle** can be understood as the associated vector bundle where $P = LM$ is the frame bundle of the tangent space with group $G =$GL$(d)$ and $V = \mathbb{R}^d$. A section $v$ of the tangent bundle is called a vector field.

The **spin bundle** is the associated bundle where $P = OLM$ is the orthonormal frame bundle of the tangent bundle, $G =$SO$(3,1)$ is the Lorentz group and $\rho$ is a spinor representation of the Lorentz group. In physics, the representation considered to express Yang-Mills theory are the Dirac bispinors transforming under the $(1/2, 0) \oplus (0, 1/2)$ representation. In that case $V = \mathbb{C}^4$. A section of the spin bundle is called a Dirac spinor.

## D.3 Connections

The notion of connection can be defined and understood in multiple ways, translating the great diversity of its application but making it a difficult concept to grasp. We will present here different definitions and discuss under which condition they are equivalent. For more, we recommend the reading of Disney-Hogg (2019).

### D.3.1 Connections on the tangent bundle: Affine and Levi-Civita connections

Vectors defined in different tangent spaces cannot be directly compared, and there exists no canonical way to define a derivative quantifying the evolution of the vectors fields on a general manifold. The notion of affine connection allows to define a derivative operator acting on vectors fields. The existence of such an operator allows to compare vectors living in different tangent spaces and define a notion of parallelism.

Let $M$ be a smooth manifold and $TM$ its tangent bundle. An **affine connection** $\nabla : v, w \to \nabla_v w$ is a bi-linear map $\Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ such that for all $f \in C^\infty(M)$ and $v, w \in \Gamma(TM)$:

- $\nabla_{fv} w = f \nabla_v w$ (linearity for the first entry)
- $\nabla_v (fw) = v(f) w + f \nabla_v w$ (Leibniz rule allowing to understand $\nabla$ as a derivative operator).

$\nabla_v w$ is understood as a derivative of the vector field $w$ in the direction given by $v$. A vector field $v$ satisfying $\nabla_v v$ is said to be **parallel transported** by $\nabla$. The curves of tangent vector $v$ are called **geodesics** of $\nabla$.

The **torsion** of a linear connection is a rank-2 tensor field acting on two vector fields $v$ and $w$ as

$$T(v, w) = \nabla_v w - \nabla_w v - [v, w] \tag{85}$$

where $[v, w](f) = v(w(f)) - w(v(f))$. Let $(M, g)$ be a (pseudo)-Riemannian manifold. The **Levi-Civita connection** is the unique affine connection satisfying:

- **metric compatibility**: $v(g(w, z)) = g(\nabla_v w, z) + g(w, \nabla_v z)$. This condition ensures the conservation of the length $g(v, v)$ of $v$ under parallel transport.
- **vanishing torsion**: $\nabla_v w - \nabla_w v = [v, w]$.

Let $e_\mu = \partial_\mu$ be a natural basis field of $TM$ over a chart $U \subseteq M$, the **components of the connection** $\Gamma^\lambda_{\ \mu\nu}$ are defined as:

$$\nabla_{e_\mu} e_\nu = \Gamma^\lambda_{\ \mu\nu} e_\lambda \tag{86}$$

The affine connection applied on two vectors $X$ and $Y$ thus become in a chart

$$\nabla_v w = (v^\nu \partial_\nu w^\lambda + \Gamma^\lambda_{\ \mu\nu} w^\mu v^\nu) e_\lambda \tag{87}$$

For the special case of the Levi-Civita connection, the connections components can be expressed in terms of the metric as

$$\Gamma^\lambda_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa} (\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) \tag{88}$$

where $\Gamma^\lambda_{\ \mu\nu} = \Gamma^\lambda_{\ \nu\mu}$ traduces the vanishing torsion.

### D.3.2 Connection on a principle bundle: Ehresmann connections

Let $P$ be a principal fiber bundle $P \xrightarrow{\pi} M$ with structure group $G$. $\mathfrak{g}$ is the Lie algebra of $G$. Let $T_pP$ be the tangent space of $P$ at the point $p$. If a point $p = (x, e) \in P$ can be understood as a choice of frame $e$ above $x \in M$, the vectors $V_p \in T_pP$ can be understood as infinitesimal changes of frames. Applying the projection $\pi : P \to M$, $\pi(p) = x$ returns the point $x \in M$ to which the frame $e$ is associated. Identically, $\pi$ can be used to obtain a vector $v \in T_xM$ associated to $x$ from a vector $V_p$ of $T_pP$ as $\pi_*(V_p) = v \in T_{\pi(p)}M$ (using the pushforward).

We define the vertical subspace $V_pP \subseteq T_pP$ at the point $p$ as

$$V_pP = \ker((\pi_*)_p) \tag{89}$$
$$= \{V^\xi \in T_pP \,/\, (\pi_*)_p(V_p) = 0\} \tag{90}$$

$V_pP$ corresponds to the set of frame transformations corresponding to "pure rotations" (i.e. translations in the fiber), which are not associated to a translation over $M$ (because their associated vector $v = \pi_*V_p$ on $TM$ is null). To each $V^\xi \in V_pP$, we can associate an element $\xi$ of $\mathfrak{g} = T_eG$ defined by its action on functions $f \in C^\infty(P)$ as

$$V_p^\xi f = \frac{\mathrm{d}}{\mathrm{d}t}(f(pe^{t\xi}))\bigg|_{t=0} = \mathbb{I}(\xi)(f) \tag{91}$$

where we defined the map $\mathbb{I} : \mathfrak{g} \to T_pP$, $\xi \to V_p^\xi$.

A **Ehresmann connection** on $P$ at the point $p$ is a choice of horizontal space $H_p \subset T_pP$ such that $T_pP = V_pP \oplus H_pP$ and satisfying $\forall h \in G, p \in P$ and $V_p \in H_pP$: $h_*V_p \in H_{ph}P$, i.e. acting with $h$ on a horizontal vector $V_p$ at point $p$ returns a horizontal vector at the point $ph$ (we say that the connection is $G$-equivariant). Once such a choice of horizontal subspace has been made, all vectors $V_p \in T_pP$ can be decomposed as $V_p = \mathrm{Ver}(V_p) + \mathrm{Hor}(V_p)$with $\mathrm{Ver}(V_p) \in V_pP$ and $\mathrm{Hor}(V_p) \in H_pP$ (A word of caution here: Even if $V_pP$ is given by $\pi$ and does not depend on the choice of connection, the value of $\mathrm{Ver}(V_p)$ and $\mathrm{Hor}(V_p)$ both depend on the choice of connection (see the illustration in Schuller (2015), class number 21)).

As such, choosing a connection allows to define a notion of "horizontality" associated to the translation of the frames over $M$ (and the associated parallel transport) by identifying locally the points of infinitesimally nearby fibers (the notion of verticality being defined naturally as the rotations of the frames on themselves at a fixed point of $M$).

A **Connection form** is a $\mathfrak{g}$ valued 1-form, $\omega : TP \to \mathfrak{g}$ satisfying

- $\forall p \in P, \omega_p : T_pP \to \mathfrak{g}$ is a vector space isomorphism.
- $\forall h \in G \,:\, h^*\omega = \mathrm{Ad}(h^{-1})\omega$.
- $\forall \xi \in \mathfrak{g}, \omega(V^\xi) = \xi$.

where the adjoint operation allows for the action of the group on itself as $\mathrm{Ad}(\tilde{h}) : G \to G, \tilde{h} \to h\tilde{h}h^{-1}$ (defining a representation of $G$).

To an Ehresmann connection, one can associate a unique connection form defined as the $\mathfrak{g}$ valued 1-form satisfying

- $\omega(V^\xi) = \xi$ if $V^\xi \in V_pP$
- $\omega(X) = 0$ if $X \in H_pP$

or alternatively, by using the map $\mathbb{I} : \mathfrak{g} \to T_pP$, $\mathbb{I}(\xi) \to V_p^\xi$ defined before, and asking:

$$\omega_p(V_p) = \mathbb{I}^{-1}(\mathrm{Ver}(V_p)) \tag{92}$$

$H_pP$ can be recovered from $\omega$ as $H_pP = \ker(\omega_p)$.

### D.3.3 Connections on an associated vector bundle: Koszul connection

The affine connections discussed in Appendix D.3.1 can be understood as a special case of a more general object: the Koszul connections. Let $\mathcal{E}$ be an associated vector bundle $\mathcal{E} = P \times_\rho V \to M$ with structure group $G$ and representation $\rho$. Let $s \in \Gamma(\mathcal{E})$ be a section of $E$ and $v \in \Gamma(TM)$ be a vector field. We define the **Koszul connection** $D_X(s)$ as the map $\Gamma(TM) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ satisfying

- $D_{fv}s = fD_v s$
- $D_v(fs) = v(f)s + fD_v s$.

Affine connections are the special case of Koszul connection for $\mathcal{E} = TM$.

Now, from the choice of a Ehresmann connection on $P$, one can define a connection form $\omega$ which itself induces a unique Koszul connection on $\mathcal{E}$. Let $e : U \to P$ a local section of $P$ (such that $\pi \circ e = \mathrm{Id}$) i.e. a choice of frames above a region $U \subset M$. A Koszul connection can be defined from $\omega$ by first considering its pullback on the base space $\omega_e = e^*\omega$, which can act on vector fields $v \in \Gamma(TM)$ as

$$\omega_e(v) = \omega(e_*(v)) \tag{93}$$

$\omega_e$ is then a $\mathfrak{g}$ valued 1-form on $M$, which can be expressed locally as $A = A^\alpha_\mu \mathrm{d}x^\mu \otimes \xi_\alpha$, where $\xi_\alpha$ is the basis (generators) of $\mathfrak{g} = T_e G$. $\omega_e$ can then act as Koszul connection on $E$ through the representation $\rho$ by introducing

$$A^i_j = \rho(\omega_e) = A^\alpha_\mu \mathrm{d}x^\mu \otimes \rho(\xi_\alpha)^j{}_i = A^\alpha{}_\mu (X_\alpha)^i_j \ \mathrm{d}x^\mu \tag{94}$$

where $X_\alpha = \rho(\xi_\alpha)$. $A$ is thus a $\mathrm{End}(E)$ valued 1-form on $M$. To obtain a Koszul connection on $\mathcal{E}$, we impose[112]

$$D_\mu e_j = A^i{}_{\mu j} e_i \tag{95}$$

with $A^i{}_{\mu j} = A^\alpha_\mu (X_\alpha)^i{}_j$. Using Leibniz rule, we can write explicitly the action of $D_v$ on a section $s = s^i e_i$ as

$$D_v s = v^\mu (\partial_\mu s^i + A^i{}_{j\mu} s^j) e_i. \tag{96}$$

Consider the special case where $P = LM$ and $E = TM$, using a chart $x$ over $U \subset M$, and the natural frame $e : U \to LM$ defined as $e = \partial_i$. $G = \mathfrak{g} = \mathrm{GL}(d, \mathbb{R})$. The connection $A = \rho(e^*\omega)$ is the special case of the affine connection discussed in Sec. D.3.1: $A^i{}_{jk} = \Gamma^i{}_{jk}$. Only in that case one can consider the torsion

$$T = \mathrm{d}\theta + \Gamma \wedge \theta \tag{97}$$

where $\theta : TM \to TM$ is the **solder form** $\theta = e_\mu \otimes \epsilon^\mu$ where $\epsilon^\mu$ is the cotangent basis associated to $e_\mu$ i.e. $\epsilon^\mu(e_\nu) = \delta^\mu_\nu$ (see 4.4 of Coquereaux (2002)). Eq. 97 can be showed to be equivalent to Eq. 85.

## D.4 Curvature

### D.4.1 For an affine connection

The **curvature** of an affine connection $\nabla$ is the $\mathrm{End}(TM)$ valued[113] two form $R$ acting on a vector field $z$ as

$$R_{v,w} z = \left( [\nabla_v, \nabla_w] - \nabla_{[v,w]} \right) z. \tag{98}$$

$R_{v,w} z$ can be understood as the holonomy acquired by $z$ if parallel transported back to itself on an infinitesimal loop constructed as a parallelogram by the vectors $v$ and $w$. In a chart it can be expressed by the tensor

$$R^\lambda{}_{\mu\rho\nu} \partial_\lambda \otimes \ \mathrm{d}x^\mu \otimes \ \mathrm{d}x^\rho \otimes \ \mathrm{d}x^\nu \tag{99}$$

[112] The link between the Ehresmann connection on $P$ and the inherited Koszul connection on $\mathcal{E}$ – as well as its geometric interpretation – can be more carefully defined using an horizontal lift (Nakahara (2003), Sec. 10.4) and/or the exterior covariant derivative on equivariant functions (Disney-Hogg 2019; Schuller 2015). Multiple equivalent ways to proceed can be considered. For example, we can consider $D_v\psi|_p = (\mathrm{d}\psi)_p(\mathrm{hor}(X^v))$, where $X^v \in T_pP$ such that $\pi_* X^v = v$ and $\psi \in C^\infty_G(P, V)$ is the representation of $s \in \mathcal{E}$ by an equivariant function on $P$.

[113] $\mathrm{End}(TM)$ is the set of all maps $TM \to TM$.

with its components being expressed in terms of the connection coefficient as

$$R^{\lambda}_{\ \mu\rho\nu} = \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\lambda}_{\nu\sigma} \tag{100}$$

This last identity can be proved as follows:

$$\begin{aligned} R^{\lambda}_{\mu\rho\nu}\partial_\lambda = R_{\partial_\rho,\partial_\nu}\partial_\mu &= ([\nabla_\rho, \nabla_\nu] - \nabla_{[\rho,\nu]})\partial_\mu \\ &= \nabla_\rho\nabla_\nu\partial_\mu - \nabla_\nu\nabla_\rho\partial_\mu - \nabla_{[\rho,\nu]}\partial_\mu \\ &= \nabla_\rho(\Gamma^{\sigma}_{\nu\mu}\partial_\sigma) - \nabla_\nu(\Gamma^{\sigma}_{\rho\mu}\partial_\sigma) - \nabla_{[\rho,\nu]}\partial_\mu \\ &= \partial_\rho\Gamma^{\sigma}_{\nu\mu}\partial_\sigma + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\kappa}_{\rho\sigma}\partial_\kappa - \partial_\nu\Gamma^{\sigma}_{\mu\rho}\partial_\sigma - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\kappa}_{\nu\sigma}\partial_\kappa \\ &= \left(\partial_\rho\Gamma^{\lambda}_{\mu\nu} - \partial_\nu\Gamma^{\lambda}_{\mu\rho} + \Gamma^{\sigma}_{\nu\mu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\lambda}_{\nu\sigma}\right)\partial_\lambda \end{aligned}$$

where in a natural basis $[\rho, \nu] = 0$.

### D.4.2 On a principal bundle

Given a Ehresmann and its associated connection form on a principal fiber bundle, there exists multiple equivalent but complementary ways to define its curvature.

Once an Ehresmann connection is defined on $P$, it is possible to define split $TP$ in an horizontal and vertical subspace. We introduce the exterior covariant derivative on $P$ as $D : \Omega^k(P) \to \Omega^{k+1}(P)$

$$D\lambda(V_1, V_2, ..., V_k) = \mathrm{d}\lambda(\mathrm{Hor}(V_1), \mathrm{Hor}(V_2), ..., \mathrm{Hor}(V_p)) \tag{101}$$

where $\lambda \in \Omega^k(P)$ is a $p$-form on $P$ and $V_1, V_2, ...V_k$ are $k$ vectors in $TP$.

The curvature $\Omega$ of the connection $\omega$ is the Lie Algebra values 2 form on $P$ ($\Omega \in \mathfrak{g} \otimes \Omega^2(P)$) defined as

$$\Omega = D\omega \tag{102}$$

Considering its action on vectors, it is possible to show that (Nakahara 2003; Schuller 2015)

$$\Omega(V, W) = \mathrm{d}\omega(V, W) + [\omega(V), \omega(W)] \tag{103}$$

for all $V, W \in TP$.

Given a choice of section $e \in \Gamma(P)$, i.e. a choice of frames over a chart $U$, it is possible to define $\Omega_e \in \mathfrak{g} \otimes \Omega^2(M)$ as

$$\Omega_e = e^*\Omega \tag{104}$$

Defining also $\omega_e = e^*\omega$, one can express

$$\Omega_e = \mathrm{d}\omega_e + [\omega_e \wedge \omega_e]_{\mathfrak{g}} \tag{105}$$

where the bracket of the Lie algebra acts as $[\omega \wedge \mu]_{\mathfrak{g}}(v, w) := [\omega(v), \mu(w)] = \omega(v)\mu(w) - \mu(w)\omega(v)$ for $\omega, \mu \in \Omega^1(M) \otimes \mathfrak{g}$ and $v, w \in \Gamma(TM)$.

## D.5 On an associated vector bundle

Similarly as in Appendix D.3.3, we can define a Koszul connection acting on $E$ associated to $P$ as $A = \rho(\omega_e)$. From this, it is possible to compute the expression of $F = \rho(\Omega_e)$ using Eq. 105 as

$$F = \mathrm{d}A + [A \wedge A]_{\rho(\mathfrak{g})} \tag{106}$$

where we absorbed all the indices but we should keep in mind that $A = A^i_{\mu j} e_i \otimes e^j \otimes \mathrm{d}x^\mu$ is a $\mathrm{End}(\mathcal{E})$ valued 1-form which take vectors on $TM$ and return a "matrix" acting on the fiber $V$ of $\mathcal{E}$. $e^i \otimes e_j$ here is a basis of $V^* \otimes V = \mathrm{End}(\mathcal{E})$ but we should keep in mind that $A$ transform in a special fashion (with a gauge transformation of the second kind) under a change of frame $e_i \to e'_i$. Similarly

$F = F^i_{j\mu\nu} e_i \otimes e^j \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is a End($\mathcal{E}$) valued 2-form which take two vectors on $TM$ and return a "matrix" acting on the fiber $V$. We thus can rewrite the previous equation more explicitly as

$$F^i_j = \mathrm{d}A^i_j + A^i_k \wedge A^k_j \tag{107}$$

or even

$$F^i_{j\mu\nu} = \partial_\mu A^i_{j\nu} - \partial_\nu A^i_{j\mu} + [A^i_{k\mu} \wedge A^k_{j\nu}] \tag{108}$$

depending on how afraid we are of the indices.

The curvature $F$ of $D$ on $\mathcal{E}$ is thus a End($\mathcal{E}$) valued two form $F = F^i_{j\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = F^\alpha_{\mu\nu} X_\alpha\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$.

Another way to proceed is to introduce the covariant exterior derivative operator $\mathrm{d}_D$. $\mathrm{d}_D$ generalizes the exterior derivative $\mathrm{d}$ to include the information of the connection. Let $D$ be a Koszul connection on $E$, $s$ be a section of $E$ ($s \in \Gamma(\mathcal{E})$) and $v \in TM$. We define the map $\mathrm{d}_D : \Omega^p(M) \otimes \Gamma(\mathcal{E}) \to \Omega^{p+1}(M) \otimes \Gamma(\mathcal{E})$ to act on $s$ as

$$d_D s(v) = D_v s \tag{109}$$

The action of $\mathrm{d}_D$ can be generalized to any $\Gamma(\mathcal{E})$ valued $p$-form as

$$d_D(s \otimes \omega) = d_D s \wedge \omega + s \otimes \mathrm{d}\omega \tag{110}$$

where $s \in \Gamma(\mathcal{E})$ and $\omega \in \Omega^p(M)$ is an arbitrary $p$-form.

$F$ can be alternatively defined from two application of $\mathrm{d}_D$ on a unit section as

$$\mathrm{d}^2_D e_j = F^i{}_j e_i \tag{111}$$

with $F^i{}_j = F^i{}_{j\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$. The two definitions can be proven equivalent as follows

$$\begin{aligned}
\mathrm{d}^2_D(e_j) &= \mathrm{d}_D(De_j) \\
&= \mathrm{d}_D(e_k A^k_j) \\
&= \mathrm{d}_D e_k \wedge A^k_j + e_k \otimes \mathrm{d}A^k_j \\
&= e_i A^i_k \wedge A^k_j + e_k\, \mathrm{d}A^k_j \\
&= e_i (A^i_k \wedge A^k_j + \mathrm{d}A^i_j) \\
&= F^i{}_j e_i
\end{aligned} \tag{112}$$

For the special case of an affine connection $A^i_{j\nu}$ is to be identified with $\Gamma^\lambda_{\mu\nu}$, this equation can be proved equivalent to Eq. 98.

# E The Bianchi identity

Using the property that $\mathrm{d}^2 = 0$, it is possible to find the relation (See Coquereaux (2002), 4.3.4):

$$\mathrm{d}F + A \wedge F = F \wedge A \tag{113}$$

This is the so-called Bianchi identity. Defining a proper way for $\mathrm{d}_D$ to act on objects as $F$, it can be simply rewritten as (see also Chap. 3 of Baez and Muniain 1994):

$$\mathrm{d}_D F = 0 \tag{114}$$

Geometrically, this equation is a general property of curvature which encodes how holonomy is propagated when combining together loops.

## F The solder form

Following Schuller (2015), let $(P, \pi, M)$ be a principal $G$-bundle and let $V$ be a vector space of same dimension as the base space $M$ on which a representation $\rho$ of $G$ can act. A **solder form** on $P$ is a one-form $\theta \in \Omega^1(P) \otimes V$ ($V$ valued 1 form on $P : TP \to V$) such that:

- $\theta$ is horizontal (on contrary of a connection form $\omega$): $\forall X \in \Gamma(TP) : \theta(\mathrm{Ver}(X)) = 0$.
- $\theta$ is $G$-equivariant: $\forall g, p \in G, P : \theta_{pg} = \rho(g^{-1})\theta_p$.
- $TM$ and $P_V = P \times_\rho V$ are isomorphic as associated bundles. $TM \underset{\text{a.b.}}{\cong} P_V$

A solder form then allows to identify $V$ with each tangent space of $M$. It does so by identifying every basis vector of $TM$ with a basis vector of $V$. Let $P = LM$ be the frame bundle of a manifold $M$, then $\theta$ is a map $\Gamma(TP) \to V = \mathbb{R}^{\dim(M)}$. Let $e$ a point in $P$ i.e. a given frame of $TM$ above a point $p = \pi(e)$ of $M$. Let $X$ be a vector field on $P$ i.e. vectors pointing in changing frame directions. We define:

$$\theta_e(X) = (u_e^{-1} \circ \pi_*)(X) \tag{115}$$

where $u_e : \mathbb{R}^{\dim(M)} \to T_pM$ such that $u_e((1, 0, 0, \cdots, 0)) = e_0$, $u_e((0, 1, 0, \cdots, 0)) = e_1 \cdots$ $u_e((0, 0, 0, \cdots, 1)) = e_{\dim(M)}$. The map $u_e^{-1} : T_pM \to \mathbb{R}^{\dim(M)}$ is identified with the coframes $\epsilon$ associated to $e$ ($\epsilon^i(e_j) = \delta^i_j$) as: $u_e^{-1}(v) = \epsilon(v) \in \mathbb{R}^{\dim(M)}$ where $v \in T_pM$ and $\epsilon(v)$ are its components in the basis $e$. The solder form $\theta$ is then $\theta_e = \epsilon \circ \pi_*$: it takes a tangent vector to the frame space and push it down on $TM$ using $\pi_*$ and then takes its components expressed in the basis $e$ at which $X$ was attached in $P$. This solder form is canonically defined and sort of "trivial", however the ability to express the components of a vector attached at $e \in P$ pushed down to $p \in M$ in the original basis $e$ at which it was attached is a unique property of the frame bundle. It allows to canonically identify the tangent spaces at every points $p$, $T_pM$ of any manifold $M$ with $\mathbb{R}^{\dim(M)}$.

Equivalently, as in Coquereaux 2002, one can make the identification between the maps from the principal bundle $P$ to $V$: $P = LM \to \mathbb{R}^{\dim(M)}$ and maps from the base space to the tangent space considered as an associated bundle: $M \to E = TM$ of fiber $V = \mathbb{R}^{\dim(M)}$. In the same way, one can identify the $V$ valued $p$-forms on $P$ as $E$ valued $p$-forms on $M$. As such $\theta$ can be instead defined as a 1-form on $M$ valued on $TM$ that is $\theta \in \mathrm{End}(TM) : TM \to TM$. One can then simply write in a given basis $e \in LM$:

$$\theta = e_\mu \otimes \epsilon^\mu \tag{116}$$

That is:

$$\theta(v) = e_\mu \otimes \epsilon^\mu(v^i e_i) = v^\mu e_\mu = v \tag{117}$$

The solder form is sometimes refereed to as the **tautological form** or the **canonical form** and plays also a fundamental role in the symplectic formulation of classical mechanics under the name of **Poincaré** or **Liouville** 1-form, which is canonically defined on the cotangent bundle $T^*M$, not on the frame bundle.

More generally, one may speak of a solder form whenever one has a $V$-valued one-form realizing an isomorphism between $TM$ and some vector bundle of fiber $V$ over $M$. In particular, the existence of a metric $g$ on $TM$ provides the musical isomorphism $\flat_g : TM \to T^*M, v \mapsto v^\flat := g(v, \cdot)$, which identifies the tangent and cotangent bundles. This is not the solder form on the frame bundle, but it is closely related to the same idea of canonically identifying tangent vectors with objects in another bundle. In coordinates, this metric-induced bundle map is $\flat_g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$, in the sense $v = v^\lambda \partial_\lambda$, one obtains the associated covector $\tilde{v} = v^\flat = g(v, \cdot) = g_{\mu\nu} v^\nu dx^\mu := \tilde{v}_\mu\, dx^\mu$.

The torsion can be defined from the solder form as the curvature can be defined from $\omega$ using $D$ or $\mathrm{d}_D$ as:

$$T = e^*(D\theta) = \mathrm{d}_D\theta = \mathrm{d}\theta + \Gamma \wedge \theta \tag{118}$$

# G Gauge theories in the language of fiber bundles

Now, we should investigate what really is about geometry within gauge theories and in what sense we can talk about geometry in the context of gauge theories. To do so, we will give a broad literature survey of the new localized gauge potential properties views, as well as their criticism and the ongoing debates. Finally, we will discuss what we can conclude from them if we take them at face value. Before we do so, we will start by some simple illustrations of the parallels between electromagnetism and the parallel transport on a surface in order to set the stage.

## G.1 Setting the stage: geometry of surfaces and electromagnetism

A 2D real vector space is equivalent (isomorphic) to a 1D complex space, we could choose equivalently to represent our vector at each point of $\mathcal{S}$ by a complex number in $\mathbb{C}$. This can be done in every tangent vector space $T_x\mathcal{S}$ under the identification $e_2 = \mathbb{i}e_1$, such that $v = (v^1 + \mathbb{i}v^2)e_1$[114]. Writing suggestively $\psi = v^1 + \mathbb{i}v^2$, we have $v = \psi(x)e_1$ where $e_1$ is a field of basis vector frame defining the orientation of the real axis of the complex plane at every point $x$ on the surface. In the identification of electromagnetism, $\psi$ is the wave-function of a charged particle or the classical counterpart of a more general relativistic quantum field. If we write the complex number in polar form $v = |\psi|e^{\mathbb{i}\phi}e_1$, were $|\psi| = \sqrt{v_1^2 + v_2^2}$ and $\phi = \mathrm{Arg}(\psi)$, we see that the vector $e_1$, by defining the direction of the real axis, sets the origin of the zero phase $\phi = 0$.

The information about the parallel transport of $v$ contained in $\Gamma^b{}_{ac}$ can be contained in a single connection coefficient $A_a$ informing about the transformation of $e_1$ in the direction $a$ which we define as

$$D_a e_1 = \mathcal{A}_a e_1, \tag{119}$$

where $\mathcal{A}_a = \Gamma^1{}_{1a} + \mathbb{i}\Gamma^2{}_{1a}$. $\mathcal{A}_a$ thus define what it means for the direction of the real axis $e_1$ to change when moving from point to point over the surface. Now, let us equip our manifold with a metric tensor $g$, allowing to compute the lengths and angles between vectors in each tangent space. Curiously, if we demand $D_a$ to be a metric preserving connection, that is to preserve the length of the vectors on the surface through parallel transport, we find that $\mathcal{A}_a$ must be a purely imaginary quantity ($\Gamma^1{}_{1a} = \Gamma^2{}_{2a} = 0$)[115]. As such, we can introduce:

$$\mathcal{A}_a = \mathbb{i}A_a \tag{120}$$

where $A_a$ is a purely real field equal to $\Gamma^2{}_{1a}$. The covariant derivative of $v$ on the surface takes the form

$$\boxed{D_a(\psi e_1) = (\partial_a + \mathbb{i}A_a)\psi e_1,} \tag{121}$$

which resemble the covariant derivative of electromagnetism Eq. (43) for a unit charge particle with $q = e = 1$. This is interesting already! By asking for the vectors length to be conserved from one vector space to another, we found that the connection must resemble the one of electromagnetism and that what we interpret as the photon $A_a$ must be a real field ($A_a \in \mathbb{R}$).

The curvature of a connection, is defined in Sec. D.4 by it's action on 3 vectors $v, w, z$ as $R_{vw}z = ([\nabla_v, \nabla_w] - \nabla_{[v,w]})z$. For the Levi-Civita connection $\nabla$, $R_{ab}e_c$ corresponds to the geometrical notion of curvature which matches our intuition. Now, let's compute the curvature associated to the connection

[114]Formally, this is done by using a so-called linear complex structure $\mathcal{J}_x : T_x\mathcal{S} \to T_x\mathcal{S}$ on each tangent vector space. $\mathcal{J}_x$, which is an operator on the tangent space acting as a multiplication by $i$, is defined such that $\mathcal{J}_x(e_1) = e_2$ and $\mathcal{J}_x(e_2) = -e_1$. Then one can identify $\mathbb{i} \in \mathbb{C}$ with $\mathcal{J}_x$ as $\mathcal{J}_x^2 = -\mathrm{Id}$. Its action on any vector is then defined as $\mathbb{i}v = \mathcal{J}_x v$, $\forall v \in T_x\mathcal{S}$. Adding this complex structure forces us to restrict our attention only on ortho-normal frames for the pair $(e_1, e_2)$, where the notion of orthonormality is defined and enforced by the complex structure $\mathcal{J}_x$, such that the two frame vectors cannot be chosen independently anymore.

[115]The proof goes as follows: taking the metric compatibility equation defined in appendix D.3 for the special case $X = \partial_a$, $Y = Z = e_1$, we find $\partial_a(g(e_1, e_1)) = g(\nabla_a e_1, e_1) + g(e_1, \nabla_a e_1) = 0$ (since $g(e_1, e_1) = 1$) so, by symmetry of the metric product $2g(\nabla_a e_1, e_1) = 2g(\mathcal{A}_a e_1, e_1) = 0$. Now, decomposing $A_a = \mathrm{Re}(\mathcal{A}_a) + \mathbb{i}\mathrm{Im}(\mathcal{A}_a)$ and recalling that $e_1 = e_1$ and $\mathbb{i}e_1 = e_2$, we fin $g(\mathrm{Re}(\mathcal{A}_a)e_1, e_1) + g(\mathrm{Im}(\mathcal{A}_a)\mathbb{i}e_1, e_1) = \mathrm{Re}(\mathcal{A}_a)g(e_1, e_1) + \mathrm{Im}(\mathcal{A}_a)g(e_2, e_1) = 0$. Since $g(e_1, e_1) = 1$ and $g(e_2, e_1) = 0$, we have $\mathrm{Re}(\mathcal{A}_a) = 0$.

$D$. By taking $X = e_a$, $Y = e_b$ and $Z = e_1$ and $[e_a, e_b] = 0$ in a natural frame, we find

$$
\begin{aligned}
R_{ab}e_1 &= D_a D_b e_1 - D_b D_a e_1 && (122)\\
&= \mathbb{i} D_a(A_b e_1) - \mathbb{i} D_b(A_a e_1) && (123)\\
&= (\mathbb{i}\partial_a A_b - A_a A_b - \mathbb{i}\partial_b A_a + A_b A_a)\, e_1 && (124)\\
&= (\partial_a A_b - \partial_b A_a)\, \mathbb{i} e_1 && (125)
\end{aligned}
$$

Defining $F$ such that $R = \mathbb{i}F$, we find

$$\boxed{F_{ab} = \partial_a A_b - \partial_b A_a} \tag{126}$$

which matches with the definition of the field strength tensor for electromagnetism (Eq. 32). So in our example the photon can be identified with as a metric preserving connection, while the strength tensor can be identified with the curvature of that connection (The factor of $\mathbb{i}$ here and in Eq. 120 correspond to the generator of U(1) associated to electromagnetism, as we will explain later).

Now, we would like the vector field $v$ to be the same no matter what choice we make regarding our coordinate system. Indeed, the arrow/geometrical object $v$ should be the same no matter what arbitrary coordinate system we use to measure its components. Naturally, we could rotate the frames arbitrarily and differently over each point of the surface as $e_1 \to e_1' = e^{\mathbb{i}\theta(x)}e_1$, and under such an arbitrary rotation of the frames, $v$ should remain the same[116]. So, noting by primes the quantities measured in a rotated frame, we should have $v = \psi e_1 = \psi' e_1'$. In order for this identity to be true, the component $\psi$ should transform in the opposite way as $\psi \to \psi' = e^{-\mathbb{i}\theta(x)}$ such that $\psi' e_1' = e^{-\mathbb{i}\theta}\psi e^{\mathbb{i}\theta} e_1 = \psi e_1$. We recognize here a gauge transformation of the first kind, which we would be tempted to interpret as the transformation of the coordinates $\psi \to \psi'$ of a vector $v$ under a frame transformation $e_1 \to e_1'$ which keeps the geometrical object $v$ invariant.

Let us now see how the coefficient the covariant derivative act on the new frame $e_1'$ and what would be the value of the coefficient of connection $A_a'$ quantifying the covariant derivative of the frame $e_1'$:

$$
\begin{aligned}
D_a e_1' = \mathbb{i} A_a' e_1' &\Rightarrow D_a(e^{\mathbb{i}\theta} e_1) = \mathbb{i} A_a' e^{\mathbb{i}\theta} e_1 && (127)\\
&\Rightarrow \frac{\partial}{\partial x^a}\left(e^{\mathbb{i}\theta}\right) e_1 + e^{\mathbb{i}\theta} D_a e_1 = \mathbb{i} A_a' e^{\mathbb{i}\theta} e_1 && (128)\\
&\Rightarrow \mathbb{i}\partial_a\theta e^{\mathbb{i}\theta} e_1 + e^{\mathbb{i}\theta}\mathbb{i} A_a e_1 = \mathbb{i} A_a' e^{\mathbb{i}\theta} e_1 && (129)\\
&\Rightarrow \mathbb{i} e^{\mathbb{i}\theta}(A_a + \partial_a\theta)\, e_1 = \mathbb{i} e^{\mathbb{i}\theta} A_a' e_1 && (130)\\
&\Rightarrow A_a' = A_a + \partial_a\theta && (131)
\end{aligned}
$$

Clearly $A_a$ transform with what can be identified as a gauge transformation of the second kind.

So overall, under an arbitrary frame rotation by an angle $\theta(x)$, $\psi$ and $A_\mu$ transform together as

$$\boxed{\begin{cases} \psi' &= e^{-\mathbb{i}\theta}\psi, \\ A_a' &= A_a + \partial_a\theta, \end{cases}} \tag{132}$$

which are exactly the transformations we encountered for gauge invariance in the previous chapter. From this analysis, we are tempted to give a geometrical interpretation of gauge invariance, as the invariance of the physical/geometrical theory under a change of the frames we use to "gauge" the theory i.e. quantifying it.

Finally, one last concept which will be crucial for the study of the Aharonov-Bohm effect is the notion of holonomy. The holonomy is the angle which would be acquired by a vector when transported parallel back to itself on a loop. Consider a loop $\gamma$ parametrized by a real parameter $\tau$ going from $\tau = 0$ to $\tau = 1$. We chose an arbitrary point $x_0$ on the loop to be the origin of $\tau = 0$. If a vector $v$ is parallel transported from the origin $\tau = 0$ all around the loop back to itself at $\tau = 1$ it might experience a rotation. With our complex structure, this means that the vector will acquire a phase as

$$v(1) = e^{\mathbb{i}\mathfrak{h}(\gamma)} v(0), \tag{133}$$

[116] We added a minus sign in the exponential, which correspond to a clockwise rotation of the frame by pure convention. The same reasoning applies identically with a positive sign.

where $v(1) = v(\tau = 1)$ and $v(0) = v(\tau = 0)$. This phase is the phase associated to the holonomy $\mathfrak{h}$. The holonomy does not depend on the choice of $x_0$ and characterizes a loop $\gamma$. The parallel transport of $v$ on the loop is characterized by the equation $D_{\gamma'} v = 0$ where $\gamma'$ is the vector field tangent to the curve $\gamma' = \frac{\partial x^a}{\partial \tau}\partial_a$. After a few steps, it is easy to show that the parallel transport of $v$ on $\gamma$ reduces to differential equation

$$\frac{\mathrm{d}\psi}{\mathrm{d}\tau} = -\mathbb{i} A_a \frac{\partial x^a}{\partial \tau}\psi \tag{134}$$

This equation can be integrated to give

$$\psi(1) = \psi(0)\exp\left(-\mathbb{i}\oint_\gamma A_a \;\mathrm{d}x^a\right) \tag{135}$$

and thus

$$\boxed{\varphi = -\oint_\gamma A_a \;\mathrm{d}x^a} \tag{136}$$

which is nothing else than the phase vector acquired in the Aharonov-Bohm effect. Here again, we can thus give a geometrical interpretation to the effect of the magnetic field on $\psi$: the phase $\psi$ acquires correspond to the rotation acquired by the vector transported around a loop of a non flat connection. Now a strong analogy can be made with the AB effect when considering the value of Eq. 136 for a loop around a cone. The holonomy around a cone is non zero while the surface of the cone on which the loop lives is essentially flat. This non vanishing holonomy is thus solely due to the apex contained in the loop which is understood as a point of non-zero curvature[117]. This is thus a direct analogy with the AB effect where the loop on which the wavefunction is considered lives in flat space surrounding the "apex" made by the solenoid at the center of the loop (for more details, see the discussion in Anandan (1995)). It is in that sense that one can consider "topological" interpretations of the AB effect and state that it comes from the fact that the base space $M$ is not simply connected due to the existence of the solenoid (Baez and Muniain 1994; Nounou 2003).

## G.2 The geometrical formulation of gauge theories using fiber bundles

Let us now use the language of fiber bundles to define a gauge theory formally. All the key concepts are defined in Sec. D.2. In full technicality: Let $(M, g)$ be the space-time, modeled as a smooth pseudo-Riemannian manifold of four dimensions. As discussed in Sec. 8.2, a gauge theory is characterized by a Lie group $G$. Let $P$ be a principal fiber bundle of structure group $G$ over $M$ named the **gauge bundle**. Let $\mathcal{E} = P \times_\rho V$ be an associated vector bundle of principal bundle $P$ and a complex vector space $V$ on which the representation $\rho$ of $G$ can act. $V$ is called the **internal space** while $E$ is called the **matter bundle**. Sections $\Psi$ of $\mathcal{E}$ are understood as matter fields. They are locally expressed as vectors of $V$ which components are defined in a basis which is identified with an element of $P$. $P$ associated to $V$ can indeed be understood as giving all possible frames (of any vector space on which a representation of $G$ can act) above each point of space-time, all related to one another by a transformation of the group $G$, called a **gauge transformation**. Locally, a point $p$ in $P$ can be represented as a doublet $p = (x, e)$ where $x$ is a point of space-time $M$ and $e$ is a group element identified with a choice of frames (We will thus abuse slightly the notation and use $p = e(x) = \{e_i(x)\}$ depending on the context). Let $\omega$ be a connection 1-form associated to an Ehresmann connection on $P$ named the **gauge connection**. This connection is a Lie-Algebra valued one form on $P$: $\omega \in \Omega^1(P) \otimes \mathfrak{g}$. $\omega$ which defines how a frame can be horizontally dragged over the base space (See Appendix D.3.2). To this connection $\omega$, one can associate a curvature $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$, which we will name the **gauge curvature**, quantifying the holonomy associated to infinitesimal loops (Appendix D.4).

Let $U \subseteq M$ be a local chart on space-time understood as a space-time region which can be mapped by coordinates $x^\mu : U \to \mathbb{R}^4$. A section of the the principal bundle $P$, $e : U \to P$ is named a **choice of gauge** which can be understood as local choice of frames. The expression of the connection in that frame is $\omega_e = e^*\omega$, expressing the parallel transport of the chosen frames $e$ over $M$. $\omega_e \in \Omega^1(M) \otimes \mathfrak{g}$

[117] For an ideal cone, the curvature is infinite at the apex. We can however imagine smoothing processes in order to render this curvature finite.

is called a **Yang-Mills field**. The curvature of $\omega$ also be expressed for the choice of frame $e$ as $\Omega_e = e^*\Omega \in \Omega^2(M) \otimes \mathfrak{g}$, expressing the rotation of a basis $e$ dragged back to itself around a loop on $M$ (holonomies) and is known as the **Yang-Mills field strength**.

A **matter field** $\Psi$ is a section of $\mathcal{E}$ expressed in $U$ as $\Psi = [e(x), \psi(x)] = \psi^i(x)e_i(x) \in \Gamma(\mathcal{E})$. Each component $\psi^i$ of $\psi$ is itself a section of the SPIN bundle[118]. The connection $\omega$ can be used to construct a unique Kozsul connection on $E$ which is known as the **gauge field** and is written as $A = \rho(\omega_e)$ (See Appendix D.3.3). $A$ is a member of $\Omega^1(M) \otimes \mathrm{End}(E)$, such that in a basis it takes the form $A = A^i{}_{\mu j} e_i \otimes e^j \otimes \, \mathrm{d}x^\mu = A^\alpha_\mu (X_\alpha)^i{}_j e_i \otimes e^j \otimes \, \mathrm{d}x^\mu$, where $(X_\alpha)^i_j = \rho(\epsilon_\alpha)^i_j$ are the representations of the generators of the Lie Algebra $\mathfrak{g}$. $A$ translates the information of the connection $\omega$ to the matter fields in $\Gamma(\mathcal{E})$ by informing about the transformation of frames. It allows to define a Koszul connection acting on matter fields $\Psi$, the gauge covariant derivative, as

$$D_v\Psi = v^\mu(\partial_\mu \psi^i + \bar{g} A^i{}_{j\mu}\psi^j)e_i. \tag{137}$$

In which an additional factor of $\bar{g}$, the gauge coupling constant, has been added to quantify the fundamental strength of the interaction through minimal coupling.

A curvature can be associated to $A$, which can be understood as the representation of the curvature of $\omega$ on $\mathcal{E}$: $F = \rho(\Omega_e)$. $F$ is a $\mathrm{End}(\mathcal{E})$ valued two form written in a basis as $F = F^i{}_{j\mu\nu} e_i \otimes e^j \otimes \, \mathrm{d}x^\mu \otimes \, \mathrm{d}x^\nu = F^\alpha_{\mu\nu}(X_\alpha)^i{}_j e_i \otimes e^j \otimes \, \mathrm{d}x^\mu \otimes \, \mathrm{d}x^\nu$, where $(X_\alpha)^i_j = \rho(\epsilon_\alpha)^i{}_j$. It can be expressed from $A$ as

$$F = \, \mathrm{d}A + \bar{g}[A \wedge A]_{\rho(\mathfrak{g})} \tag{138}$$

The various terms in the Lagrangian $\mathcal{L}_{\psi A}$ (Eq. 48) can be rewritten in a more geometric "coordinate free" way as

$$\frac{1}{4} F^\alpha_{\mu\nu} F^{\mu\nu}_\alpha \, \mathrm{d}^4x = \frac{1}{2} F \wedge \star F \tag{139}$$

and

$$\bar{\psi}(\mathbb{i}\gamma^\mu D_\mu - m)\psi = \langle \Psi | (\mathbb{i}\not{D} - m)\Psi \rangle_V \tag{140}$$

where $\not{D} = \gamma^\mu(\partial_\mu + \bar{g} A^\alpha_\mu X_\alpha)$ is the Dirac operator, $\langle.|.\rangle_V$ is the Dirac product $\langle \Psi_1 | \Psi_2 \rangle_V = \psi_1^\dagger \gamma^0 \psi_2$. The equations of motion can similarly be written

$$\begin{cases} (\mathbb{i}\not{D} - m)\Psi = 0 \\ \star \, \mathrm{d}_D \star F = \bar{g}\mathcal{J} \end{cases} \tag{141}$$

There are now two ways to think of gauge invariance in this context. As we did suggestively in Sec. G.1, we can consider that gauge invariance is simply the invariance of the equation of motions and Lagrangian under a change of frame. This invariance is intrinsically encoded in $\mathcal{E}$ through its defining equivalence relation (see Appendix D.2). Considering $h = h(x)$ as a section of $P$, a gauge transformation would then be a transformation $\Psi = [ph, \rho(h^{-1})\psi] = [p, \psi]$, leaving the geometrical object $\Psi$ unchanged. The connection $A$ would remain unchanged by such a transformation and defined by its action on the representative of the equivalence class $\Psi = [p, \psi] = \psi^i e_i$.

Now, the equivalence relation defining $\mathcal{E}$ implies that $[p, \rho(h)\psi] = [hp, \rho(h^{-1})\rho(h)\psi] = [hp, \rho(h)\psi]$. Using this fact, we can also define a gauge transformations as the joint transformations

$$\begin{cases} \Psi \to \Psi' = [ph, \psi] = [p, \rho(h)\psi] \\ A \to A' = hAh^{-1} - h^{-1} \, \mathrm{d}h \end{cases} \tag{142}$$

in which we actively change the frame $p' = ph$ along with the components of $A$ in order to express the covariant derivative in the horizontal space in $T_{ph}P$ of the principal bundle instead of $T_pP$ (see Sec. D.3.2). This transformation is called a local vertical bundle automorphism of $P$ and also leaves the equations of motion of the theory invariant. This way of reading gauge transformation is treating gauge as an active transformation and is not a frame transformation.

[118] Formally $\psi$ is not only a section of $\mathcal{E}$ but a section of $\mathcal{E} \otimes S$, where S is the spin bundle. Thus $\psi$ are spinors and takes values in $V \otimes \mathbb{C}^4$. The components in $V$ are associated to $P$ and the components in $\mathbb{C}^4$ are associated to the frame bundle of the tangent space $LM$.

The full standard model of particle physics (without the Higgs mechanism) is represented by the gauge group $G$ =U(1)×SU(2)× SU(3) acting on vector spaces with different representations $\rho$ depending on the matter type under consideration. Actually, all matter fields are coupled to the same principal bundle of frames, but in different associated bundles having different representations. These representations are called the **charges** of the particles and explain the fact that different particles can couple differently to different forces.

In the fiber bundle language, general relativity has a very similar structure as gauge theories, in which the (orthonormal) frame bundle plays the role of the principal bundle $P$ and the tangent bundle or the spin bundle plays the role of the associated bundle $\mathcal{E}$. We will further discuss these similarities and possible differences in Appendix B.

# H Wu and Yang's argument against $B$ interpretations

We reproduce here the arguments of Wu and Yang 1975b. See also the discussion in Healey 2007. Consider two different SU(2) gauge potentials $(A^\alpha_\mu)_A$ and $(A^\alpha_\mu)_B$. Let's consider the first potential to be defined in a given coordinate frame $x^\mu$ as

$$(A^\alpha_0)_A = 0 \qquad (A^\alpha_a)_A = \epsilon^\alpha_{\ a\gamma}\frac{x^\gamma}{r^2}(\Phi - 1) \tag{143}$$

where $\alpha = 1,2,3$ runs over the generators of SU(2) and $a = 1,2,3$ is a space index. $\epsilon$ is the Levi-Civita tensor and $\Phi$ is a constant. For simplicity, we write the structure constant of SU(2) as $f^\alpha_{\beta\gamma} = \epsilon^\alpha_{\beta\gamma}$ and set $\bar{g} = 1$. As such, the only non-vanishing component of the strength tensor are

$$(F^\alpha_{ab})_A = \partial_a(A^\alpha_b)_A - \partial_b(A^\alpha_a)_A + \epsilon^\alpha_{\ \beta\gamma}(A^\beta_a)_A(A^\gamma_b)_A \tag{144}$$

$$= \epsilon_{abc}\frac{x^c x^\alpha}{r^4}(1 - \Phi^2) \tag{145}$$

The current associated to this case can be computed from $F$ using the Yang-Mills equation. While $(J^\alpha_0)_A = 0$,

$$(J^\alpha_b)_A = -\partial^a(F^\alpha_{ab})_A - f^\alpha_{\ \beta\gamma}(A^\beta_a)_A(F^\gamma_{ab})_A \tag{146}$$

$$= \epsilon^\alpha_{\ cb}\frac{x^c}{r^4}\Phi(\Phi^2 - 1) \tag{147}$$

Thus, if $\Phi \neq \{0, \pm 1\}$, these equations do not describe a sourceless case.

Now consider the potential $(A^\alpha_\mu)_B$ which have vanishing components for all of its coordinates except the spherical coordinate $\phi$ such that

$$(A^3_\phi)_B = \frac{-(1 - \Phi^2)}{r}\arctan(\theta) \tag{148}$$

with corresponding strength tensor's non vanishing terms

$$(F^3_{ab})_B = \epsilon_{abc}\frac{x^c}{r^3}(1 - \Phi^2). \tag{149}$$

Computing the current associated to this strength tensor demonstrates that it is sourceless. Under a gauge transformation rotating the direction 3 in the isospin direction $(1/r)(x^1, x^2, x^3)$, $(F^3_{ab})_B$ takes the same form as $(F^3_{ab})_A$, such that both $(A^\alpha_\mu)_A$ and $(A^\alpha_\mu)_B$ share the same strength tensor but describe a different physical situation.

The reason for the existence of this effect in non Abelian theories while it is absent in Abelian ones comes from the distinction between Maxwell's equations ($\partial_\mu F^{\mu\nu} = J^\nu$) and Yang-Mills equations ($\partial_\mu F^{\mu\nu}_\alpha + f^\gamma_{\ \alpha\beta}A^\beta_\mu F^{\mu\nu}_\gamma = J^\nu_\alpha$). Yang-Mills equations relate the gauge potential and the current through the covariant derivative of $F^{\mu\nu}_\alpha$, such that a change in the source $J^\mu_\alpha$ can thus be balanced by a change in the potentials $A^\alpha_\mu$.